Budgeted Opinion Dynamics with Varying Susceptibility to Persuasion

Final Report

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Abstract

The Opinion Susceptibility Problem is an optimization problem in opinion dynamics. It explores the optimal way to guide the aggregate opinions in a social network to an extreme by modifying the agents’ susceptibility to persuasion. Based on the opinion dynamics model proposed by Abebe et al. [1], the L1-budgeted variant of the problem is defined and analysed in this project. This variant imposes an upper bound on the sum of the absolute changes of the agents’ resistance parameters. In this project, the NP-hardness of the L1-budgeted variant is established through a reduction to the vertex cover problem for regular graphs. Moreover, by analysing the possible locations of the optimal solution, the Projected Gradient Algorithm is constructed and implemented to search for the optimum within a confined search space on the feasible region boundary. The algorithm employs the Optimistic Local Search strategy, originally proposed in Chan et al. [2] to solve the unbudgeted variant of the problem, to locate a starting point. Moreover, the objective and the gradient approximations by iterative processes are used to enhance the efficient of the algorithm. From the algorithm experiments, it is found that for almost all agents, the budget allocated to each agent is either zero or is adequate to reach his upper/lower bound of resistance, meaning that the optimal point found is usually a vertex of the feasible region that uses up the budget.
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1 Introduction

The Opinion Susceptibility Problem is an optimization problem related to opinion dynamics. It tries to address how to guide the public towards a particular point of view on a social issue via the intervention of people’s resistance to change their opinions. This project will approach the L1-budgeted variant of this problem mathematically and algorithmically.

1.1 Background

The process of social influence is a significant basis for opinion formation, decision making, and the shaping of an individual’s identity [3]. What is more, it drives every social phenomenon ranging from the emergence of trends, diffusion of rumour, and the shaping of public views about social issues [4][5]. However, given the intrinsically tremendous number of components and their complex relationships in a social network, the opinion-evolving mechanism remains poorly understood [4].

Opinion dynamics emerged from various disciplines in social science. It investigates how the opinions of interrelated individuals diffuse and evolve dynamically in a social network using mathematical models and computational tools [5][6]. Based on the specific opinion representation, interaction rules, and the social structure that relates the agents in real life, a wide diversity of opinion dynamics models are available for simulating the dynamics in different social settings [5]. As suggested in [5], the most popular modelling method is agent-based modelling. This method captures the opinion drifts of each individual using mathematical equations. The agent-based models serve as a laboratory in which different factors are controllable, allowing the discovery of theories that explain phenomena and make real-life predictions about social opinions [7].

In 2018, Abebe et al. [1] proposed an agent-based opinion dynamics model based on the works of DeGroot [8] and Friedkin and Johnsen [3]. This model describes a discrete-time iterative process of opinion evolution where the opinion update of the agents depends on their susceptibility to persuasion, i.e. their willingness to modify their views. Under this model, Abebe et al. [1] further proposed the Opinion Susceptibility Problem. This problem asks how to optimally guide the public towards a particular point of view on a social issue via the intervention of people’s resistance to persuasion. Since opinion interference is widely applicable in advertising and ideology spread [1], and people’s susceptibility to persuasion can be intervened by, for example, persuasive cues [9], it is worthwhile to consider the Opinion Susceptibility Problem.

This original problem, known as the unbudgeted variant, was analysed in Chan et al. [2] and was found solvable in polynomial time using local search techniques. Also, there is an $L0$-budgeted variant of the problem. In this problem, the number of agents whose resistance parameter is to be changed cannot exceed a given bound. This problem with the target-set constraint is proven to be NP-hard by Abebe et al. [1].
1.2 Objectives

This project will focus on the *L1-budgeted variant* of the problem, in which the L1-norm of the perturbation of the resistance vector cannot exceed a given budget. This variant addresses the concern that the cost of intervention increases with the total resistance change.

In view of the previous works on the unbudgeted variant and the L0-budgeted variant, this project aims to investigate whether the L1-budgeted problem is NP-hard. Also, an efficient algorithm will be designed to either optimally solve or give an approximated solution to the L1-budgeted problem, depending on the hardness of the problem. The algorithm will be implemented and tested by running experiments.

1.3 Outline

As follows, this report will erst introduce the opinion dynamics model and the definition of the variants of the Opinion Susceptibility Problem in Section 2. Then, Section 3 will offer proof of the NP-hardness of the L1-budgeted variant. Next, Section 4 will provide some proven properties of the objective function, which are essential for discussion on the search space in Section 5 and the algorithm proposed in Section 6. After that, the experimental results of the algorithm will be presented and analysed in Section 7. Finally, Section 8 will summarize the whole progress report.
2 Preliminaries

This section will first give the details of the opinion dynamics model on which the Opinion Susceptibility Problem is defined. Then, the definitions of the unbudgeted variant and the L1-budgeted variant of the problem will be introduced.

2.1 Opinion dynamics model studied

This project places its focus on the opinion dynamics model considered in [1] and [2]. Let \([n]\) be the set of agents in the model, where \([n]\) denotes the set \(\{1, \ldots, n\}\). Their innate opinions are captured by a vector \(s := [s_i] \in [0, 1]^n\). The interaction matrix \(P := [P_{ij}] \in [0, 1]^{n \times n}\) captures the relationships between agents. In particular, its entry \(P_{ij}\) is the relative social influence of agent \(j\) on agent \(i\). This matrix \(P\) is row-stochastic, i.e. each entry of \(P\) is nonnegative, and every row sums to 1. Each agent \(i\)'s susceptibility to persuasion is measured by the resistance parameter \(\alpha_i \in [0, 1]\), where a higher \(\alpha_i\) value suggests agent \(i\) is more opposed to persuasion. For each agent \(i\), a lower bound \(l_i\) and an upper bound \(u_i\) of the resistance parameter \(\alpha_i\) are assumed to be intrinsically fixed.

The opinion dynamics evolves in discrete time according to the initial condition \(z^{(0)} = s\) and the non-homogeneous first-order matrix difference equation

\[
z^{(t+1)} := As + (I - A)Pz^{(t)},
\]

where \(A = \text{Diag}(\alpha)\) is the diagonal matrix with \(A_{ii} = \alpha_i\) and \(I\) is the identity matrix. In other words, for any time \(t\), the opinion of agent \(i\) at time \(t + 1\), given by \(z^{(t+1)}_i := \alpha_is_i + (1 - \alpha_i)\sum_{j \in [n]} P_{ij}z^{(t)}_j\), is a convex combination of his innate opinion \(s_i\) and the weighted opinion average of all the agents at time \(t\) according to the relative social influence on agent \(i\) of all other agents. Setting \(z^{(t+1)} = z^{(t)}\) yields the opinion vector at the equilibrium state

\[
z = [I - (I - A)P]^{-1}As.
\]

Note that the inverse \(M := [I - (I - A)P]^{-1}\) exists under the assumption that \(\alpha \in (0, 1)^n\) and that \(P\) arises from a connected graph [2].

2.2 Opinion Susceptibility Problem and its variants

Under this model, the Opinion Susceptibility Problem aims to choose a resistance vector \(\alpha\) such that the sum of the equilibrium opinions \(\sum_{i \in [n]} z_i = 1^Tz\) is optimized. The project will focus on the minimization problem; any maximization problem can undergo an opinion transformation \(x \mapsto 1 - x\) to an equivalent minimization problem [1][2].

Definition 2.1 (Opinion Susceptibility Problem). Given a set \([n]\) of agents with innate opinions \(s \in [0, 1]^n\) and interaction matrix \(P \in [0, 1]^{n \times n}\), suppose for each agent \(i \in [n]\), its resistance is restricted to some interval \([l_i, u_i] \subseteq [0, 1]\) where \(0 \leq l_i \leq u_i \leq 1\). Define \(z^{(0)} = s\) and \(z^{(t+1)} := As + (I - A)Pz^{(t)}\) for any \(t \geq 0\), where \(A = \text{Diag}(\alpha)\) is the diagonal matrix with \(A_{ii} = \alpha_i\). The objective is to choose
\[ \alpha \in \prod_{i \in [n]} [l_i, u_i] \subseteq [0, 1]^n \] such that the following objective function is minimized:

\[ f(\alpha) := 1^T \lim_{t \to \infty} z^{(t)}. \]

If \( M := [I - (I - A)P]^{-1} \) exists, then \( \lim_{t \to \infty} z^{(t)} = z = MAs \) and thus the objective function is equal to:

\[ f(\alpha) = 1^T MAs. \]

The problem defined in Definition 2.1 is known as the unbudgeted variant. This project considers the L1-budgeted variant of the problem: given some initial resistance vector \( \alpha^{(0)} \) and a perturbation budget \( b \), its L1-normed distance from the resistance vector chosen for objective minimization cannot exceed \( b \).

**Definition 2.2** (L1-Budgeted Variant of the Opinion Susceptibility Problem). Given a set \([n]\) of agents with innate opinions \( s \in [0, 1]^n \) and interaction matrix \( P \in [0, 1]^{n \times n} \), suppose for each agent \( i \in [n] \), its resistance is restricted to some interval \( [l_i, u_i] \subseteq [0, 1] \) where \( 0 \leq l_i \leq u_i \leq 1 \), and its initial resistance is given by \( \alpha_i^{(0)} \in [l_i, u_i] \). Define \( z^{(0)} = s \) and \( z^{(t+1)} := As + (I - A)Pz^{(t)} \) for any \( t \geq 0 \), where \( A = \text{Diag}(\alpha) \) is the diagonal matrix with \( A_{ii} = \alpha_i \). With a perturbation budget \( b > 0 \) provided, the objective is to choose \( \alpha \in \prod_{i \in [n]} [l_i, u_i] \subseteq [0, 1]^n \) such that the following objective function is minimized:

\[ f(\alpha) := 1^T \lim_{t \to \infty} z^{(t)} \]

subject to the budget constraint \( \| \alpha - \alpha^{(0)} \|_1 \leq b \). Here, \( \| x \|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \) denotes the Lp-norm of the vector \( x = [x_i] \in \mathbb{R}^n \).

If \( M := [I - (I - A)P]^{-1} \) exists, then \( \lim_{t \to \infty} z^{(t)} = z = MAs \) and thus the objective function is equal to:

\[ f(\alpha) = 1^T MAs. \]

Although this project places its focus on the L1-budgeted problem, the previous works on the unbudgeted problem in Chan et al. [2] will be helpful in solving the L1-budgeted problem in multiple ways.
3 NP-hard L1-Budgeted Problem

In this section, the NP-hardness of the L1-budgeted Opinion Susceptibility Problem will be proven. Inspired by the proof of NP-hardness of the L0-budgeted problem in [1], the proof in this section will show that the vertex cover problem for regular graphs is polynomial-time reducible to the L1-budgeted problem.

To be more precise, any instance of vertex cover problem for a $d$-regular graph with $n$ vertices can be mapped to an instance of the decision version of the L1-budgeted problem with a set of $n + 1$ agents labelled $\{0\} \cup [n]$ in polynomial time. Every agent in the opinion dynamics model has an innate opinion of value 0 and initial resistance 0 except agent 0, who has an innate opinion of value 1 and initial resistance 1. The resistance parameter of every agent can be set to any value between 0 and 1 inclusive except that of agent 0, which is fixed at 1. Therefore, the available budget will not be assigned to agent 0.

The interaction matrix of the L1-budgeted problem will be a convex combination of the row-stochastic matrix that corresponds to the unweighted complete graph on the set of agents $\{0\} \cup [n]$, as well as the row-stochastic matrix that corresponds to the union of a self-loop on agent 0 and the unweighted $d$-regular graph on $[n]$. The ratio of the combination is carefully chosen so that the sum of equilibrium opinions of the agents in the opinion dynamics model will exceed a certain threshold if the $d$-regular graph does not have a vertex cover of size $k$. This section will explain the selection of the ratio to pave the way for introducing Theorem 3.15, which establishes the NP-hardness of the problem.

To begin with, the definitions of an $M$-matrix and a diagonally dominant matrix are given below.

**Definition 3.1 (M-matrix).** A matrix $A$ is called an $M$-matrix if it can be expressed in the form $A = sI - B$ with $s \geq 0$, $B \geq 0$ and $s \geq \rho(B)$, where $\rho(B)$ denotes the Perron root of $B$.

**Definition 3.2 (Diagonally dominant).** A matrix $A = [a_{ij}]$ is said to be diagonally dominant if $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ for all $i$.

The inverse of any nonsingular $M$-matrix, if exists, is nonnegative. A proof of this result can be found in [10]. The next lemma will be used for proving that the matrix $I - (I - A)P$ in Definition 2.1 is an $M$-matrix. Then, its inverse $M := [I - (I - A)P]^{-1}$, if exists, is nonnegative.

**Lemma 3.3 (Sufficient condition for M-matrices [10]).** If $A = [a_{ij}]$ is diagonally dominant with $a_{ii} > 0$ and $a_{ij} \leq 0$ for $i \neq j$, then $A$ is an $M$-matrix.

The following lemma will be used for simplifying the computation which involves the matrix inverse $M$. Denote the $(i, j)$-entry of $M$ by $M_{ij}$ for any $i, j \in [n]$. Compared to the original version of the lemma in Chan et al. [2], this modified version is weaker in the sense that the inequalities about $M_{kk}$ and $M_{kj}$ are no longer strict, but it no longer requires that $0 < \alpha_i < 1$ for any $i$ or $P$ is irreducible. This version below is applicable as long as $P_{ii} \neq 1$ for all $i$. 

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Lemma 3.4 (Modified Version of Inverse Arithmetic [2]). Given $\alpha \in [0, 1]^n$, let $A := \text{Diag}(\alpha)$ and $P$ be a row-stochastic matrix with $P_{ii} \neq 1$ for each $i \in [n]$. Define $X := I - (I-A)P$. Then $X$ is a diagonally dominant $M$-matrix. Moreover, if the inverse $M := X^{-1}$ exists, then $M_{kk} \geq 1$ and $M_{kj} \geq 0$ for each $j \neq k$. Also, if $\alpha_k \neq 1$, then $(PM)_{kk} = \frac{M_{kk} - 1}{1 - \alpha_k}$ and $(PM)_{kj} = \frac{M_{kj}}{1 - \alpha_k}$ for any $j \neq k$.

Proof. In view of Lemma 3.3, to show that $X$ is a diagonally dominant $M$-matrix, it suffices to show that $X$ is diagonally dominant with $X_{ii} > 0$ and $X_{ij} \leq 0$ for $i \neq j$. Indeed, since

$$X_{ii} = 1 - (1 - \alpha_i)P_{ii} > 1 - (1 - \alpha_i) = \alpha_i \geq 0$$

for any $i$, and

$$X_{ij} = -(1 - \alpha_i)P_{ij} \leq 0$$

for any $i \neq 0$,

we have

$$\sum_{j \neq i} |X_{ij}| = \sum_{j \neq i} (1 - \alpha_i)P_{ij} = (1 - \alpha_i)(1 - P_{ii}) = 1 - \alpha_i - (1 - \alpha_i)P_{ii} \leq 1 - (1 - \alpha_i)P_{ii} = |X_{ii}|$$

for any $i$,

so $X$ is a diagonally dominant $M$-matrix. This implies $M = X^{-1}$ is a nonnegative matrix.

By the definition of $M$, we have $[I - (I-A)P]^\top M = I$. For any $j \neq k$, by considering the $(k,k)$-th entry and the $(k,j)$-th entry respectively, we obtain

$$M_{kk} - \sum_{i\in [n]} (1 - \alpha_k)P_{ki}M_{ik} = 1$$

and

$$M_{kj} - \sum_{i\in [n]} (1 - \alpha_k)P_{ki}M_{ij} = 0.$$

Hence, we have, for any $j \neq k$,

$$M_{kk} = 1 + \sum_{i\in [n]} (1 - \alpha_k)P_{ki}M_{ik} \geq 1$$

and

$$M_{kj} = \sum_{i\in [n]} (1 - \alpha_k)P_{ki}M_{ij} \geq 0.$$

Moreover, if $\alpha_k \neq 1$, then

$$(PM)_{kk} = \sum_{i\in [n]} P_{ki}M_{ik} = \frac{M_{kk} - 1}{1 - \alpha_k}$$

and

$$(PM)_{kj} = \sum_{i\in [n]} P_{ki}M_{ij} = \frac{M_{kj}}{1 - \alpha_k}$$

for any $j \neq k$. 

The following lemma, first given in [2], provides a simple formula for the partial derivatives of $f$ with respect to $\alpha_i$. It will be used for calculating the second-order partial derivatives of $f$.

Lemma 3.5 (Partial derivative of $f$ with respect to $\alpha_i$ [2]). Given $\alpha \in [0, 1]^n$, let $A := \text{Diag}(\alpha)$, $s$ be an innate opinion vector and $P$ be an interaction matrix. Denote $M := [I - (I - A)P]^{-1}$. Recall that $z(\alpha) := MA$ is the equilibrium opinion vector. For any $i \in [n]$, the partial derivative of $f$ with respect to $\alpha_i$ is given by

$$\frac{\partial f}{\partial \alpha_i} = \frac{s_i - z_i(\alpha)}{1 - \alpha_i} \cdot 1^T Me_i.$$

The following lemma will be important for understanding the optimal assignment of budget when the given $d$-regular graph has no vertex cover of size $k$. Specifically, fixing the resistance parameters of all but two agents, the lemma provides an insight into the optimal way to allocate the budget to the two agents.
Lemma 3.6. Given a set of $n+1$ agents labelled \{0\} \cup [n], suppose $\alpha \in [0,1]^{n+1}$ with $\alpha_0 = 1$, let $A := \text{Diag}(\alpha)$. Set $s_i = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$. Let $P \in \mathbb{R}^{(n+1) \times (n+1)}$ be an interaction matrix. Assume that $M := [I - (I - A)P]^{-1}$ exists. For any distinct $i, j \in [n]$, the second directional derivative of $f$ at $\alpha$ in the direction $e_i - e_j$ is given by

$$(e_i - e_j)^T H f(\alpha)(e_i - e_j) = \frac{2}{(1 - \alpha_i)(1 - \alpha_j)} \cdot \left\{ z_i(\alpha) y_{ij}(\alpha) + z_j(\alpha) y_{ji}(\alpha) \right\}.$$ 

where $y_{ij}(\alpha) := 1^T M e_i \cdot (M_{jj} - 1) - 1^T M e_j \cdot M_{ji}$ for any distinct $i, j \in [n]$.

Proof. By the definition of the inverse of a matrix $B$, we have $BB^{-1} = I$. The partial derivative with respect to a variable $t$ is: $\frac{\partial B}{\partial t} B^{-1} + B \frac{\partial B^{-1}}{\partial t} = 0$. Hence, we have $\frac{\partial B^{-1}}{\partial t} = -B^{-1} \frac{\partial B}{\partial t} B^{-1}$. Applying the above result with $B = I - (I - A)P$ and $t = \alpha_i$, we get $\frac{\partial M}{\partial \alpha_i} = -Me_i e_i^T PM$. Combining with the results in Lemmas 3.4 and 3.5, we obtain the second-order partial derivatives of $f$ as follows:

$$\frac{\partial^2 f(\alpha)}{\partial \alpha_i^2} = \frac{s_i - z_i(\alpha)}{1 - \alpha_i} 1^T \frac{\partial M}{\partial \alpha_i} e_i + \frac{s_i - z_i(\alpha)}{(1 - \alpha_i)^2} 1^T M e_i - \frac{1}{1 - \alpha_i} 1^T M e_i \frac{\partial z_i(\alpha)}{\partial \alpha_i}$$

$$= -\frac{s_i - z_i(\alpha)}{1 - \alpha_i} 1^T M e_i e_i^T PM e_i + \frac{s_i - z_i(\alpha)}{(1 - \alpha_i)^2} 1^T M e_i = -\frac{s_i - z_i(\alpha)}{(1 - \alpha_i)^2} 1^T M e_i \cdot (M_{ii} - 1) + \frac{s_i - z_i(\alpha)}{(1 - \alpha_i)^2} 1^T M e_i - \frac{s_i - z_i(\alpha)}{(1 - \alpha_i)^2} 1^T M e_i \cdot M_{ii}$$

$$= -2 \frac{s_i - z_i(\alpha)}{(1 - \alpha_i)^2} 1^T M e_i \cdot (M_{ii} - 1),$$

$$\frac{\partial^2 f(\alpha)}{\partial \alpha_i \partial \alpha_j} = \frac{s_i - z_i(\alpha)}{1 - \alpha_i} 1^T \frac{\partial M}{\partial \alpha_j} e_i - \frac{1}{1 - \alpha_i} 1^T M e_i \frac{\partial z_i(\alpha)}{\partial \alpha_j}$$

$$= -\frac{s_i - z_i(\alpha)}{1 - \alpha_i} 1^T M e_i e_j^T PM e_i - \frac{s_i - z_i(\alpha)}{(1 - \alpha_i)(1 - \alpha_j)} 1^T M e_i e_j^T M e_j$$

$$= -\frac{s_i - z_i(\alpha)}{(1 - \alpha_i)(1 - \alpha_j)} 1^T M e_i \cdot M_{jj} - \frac{s_i - z_i(\alpha)}{(1 - \alpha_i)(1 - \alpha_j)} 1^T M e_i \cdot M_{ij}$$

for any distinct $i, j \in \{0\} \cup [n]$.
Denote by $\mathbf{H}_f$ the Hessian matrix of the scalar-valued function $f$. Since $s_i = 0$ for any $i \in [n]$, the second directional derivative of $f$ at $\alpha$ in the direction $e_i - e_j$, where $i, j \in [n]$ are distinct, is given by

$$(e_i - e_j)^T \mathbf{H}_f(\alpha)(e_i - e_j)$$

\[= \frac{\partial^2 f(\alpha)}{\partial \alpha_i^2} + \frac{\partial^2 f(\alpha)}{\partial \alpha_j^2} - \frac{\partial^2 f(\alpha)}{\partial \alpha_i \partial \alpha_j} - \frac{\partial^2 f(\alpha)}{\partial \alpha_j \partial \alpha_i} \]

\[= 2 \cdot \frac{z_i(\alpha)}{(1 - \alpha_i)(1 - \alpha_j)} \mathbf{1}^T \mathbf{e}_i \cdot (\mathbf{M}_{ii} - 1) + 2 \cdot \frac{z_j(\alpha)}{(1 - \alpha_i)(1 - \alpha_j)} \mathbf{1}^T \mathbf{e}_j \cdot (\mathbf{M}_{jj} - 1) -
\]

\[2 \cdot \frac{z_i(\alpha)}{(1 - \alpha_i)(1 - \alpha_j)} \mathbf{1}^T \mathbf{e}_j \cdot \mathbf{M}_{ii} - 2 \cdot \frac{z_j(\alpha)}{(1 - \alpha_i)(1 - \alpha_j)} \mathbf{1}^T \mathbf{e}_i \cdot \mathbf{M}_{jj}.
\]

Since $\alpha$ is a stationary point on $S$, the first-order directional derivative of $f$ at $\alpha$ in the direction $e_i - e_j$ is

$$(e_i - e_j)^T \nabla f(\alpha) = \frac{\partial f(\alpha)}{\partial \alpha_i} - \frac{\partial f(\alpha)}{\partial \alpha_j} = \frac{z_i(\alpha)}{1 - \alpha_i} \mathbf{1}^T \mathbf{e}_i + \frac{z_j(\alpha)}{1 - \alpha_j} \mathbf{1}^T \mathbf{e}_j = 0,$$

which gives

$$\frac{z_i(\alpha)}{1 - \alpha_i} \mathbf{1}^T \mathbf{e}_i = \frac{z_j(\alpha)}{1 - \alpha_j} \mathbf{1}^T \mathbf{e}_j.$$ 

Thus, we have

$$(e_i - e_j)^T \mathbf{H}_f(\alpha)(e_i - e_j)$$

\[= 2 \cdot \frac{z_i(\alpha)}{(1 - \alpha_i)(1 - \alpha_j)} \mathbf{1}^T \mathbf{e}_j \cdot (\mathbf{M}_{ii} - 1) + 2 \cdot \frac{z_j(\alpha)}{(1 - \alpha_i)(1 - \alpha_j)} \mathbf{1}^T \mathbf{e}_j \cdot (\mathbf{M}_{jj} - 1) -
\]

\[2 \cdot \frac{z_i(\alpha)}{(1 - \alpha_i)(1 - \alpha_j)} \mathbf{1}^T \mathbf{e}_j \cdot \mathbf{M}_{ii} - 2 \cdot \frac{z_j(\alpha)}{(1 - \alpha_i)(1 - \alpha_j)} \mathbf{1}^T \mathbf{e}_i \cdot \mathbf{M}_{jj}.
\]

\[= \frac{2}{(1 - \alpha_i)(1 - \alpha_j)} \cdot \left\{ z_i(\alpha) \left[ \mathbf{1}^T \mathbf{e}_i \cdot (\mathbf{M}_{jj} - 1) - \mathbf{1}^T \mathbf{e}_j \cdot \mathbf{M}_{ji} \right] +
\]

\[z_j(\alpha) \left[ \mathbf{1}^T \mathbf{e}_j \cdot (\mathbf{M}_{ii} - 1) - \mathbf{1}^T \mathbf{e}_i \cdot \mathbf{M}_{ij} \right] \right\} \]

\[= \frac{2}{(1 - \alpha_i)(1 - \alpha_j)} \cdot \left[ z_i(\alpha) y_{ij}(\alpha) + z_j(\alpha) y_{ji}(\alpha) \right].
\]

as desired. \qed

Fix an arbitrary $\alpha$ with $\alpha_0 = 0$ and $0 < \alpha_i, \alpha_j < 1$, if $z_i(\alpha)$ and $z_j(\alpha)$ are strictly positive whereas $y_{ij}(\alpha)$ and $y_{ji}(\alpha)$ are strictly negative, then Lemma 3.6 suggests that the second-order directional derivative of $f$ in the direction $e_i - e_j$, evaluated at $\alpha$, will be negative. This also means that $\alpha$ cannot be a strict minimum point because if so, the first-order directional derivative of $f$ in the direction $e_i - e_j$ evaluated at $\alpha$ must be zero, and the negativity of the second-order directional derivative suggests that $\alpha$ is indeed a strict local maximum point in the direction $e_i - e_j$, which is a contradiction.
Suppose it is true that $z_i(\alpha), z_j(\alpha) > 0$ and $y_{ij}(\alpha), y_{ji}(\alpha) < 0$ for any $\alpha$ with $\alpha_0 = 0$ and $0 < \alpha_i, \alpha_j < 1$. Let $b = \alpha_i + \alpha_j$. Fix all $\alpha_k$'s except $\alpha_i$ and $\alpha_j$. Consider the two redistributions of the budget $b$ among the two agents in which one of them does not receive any budget unless the other agent receives 1 unit of budget. By the above argument, one of these redistribution must be an optimal budget assignment. Applying this argument repeatedly to each pair of agents. Then if the available budget is $k$ units where $k \in \mathbb{N}$, then there exists an optimal assignment such that exactly $k$ agents receive a unit of budget.

To guarantee that $y_{ij}(\alpha) < 0$ always holds for any distinct $i, j \in [n]$ regardless of the resistance vector $\alpha$, the interaction matrix $P$ should be picked carefully. Thus, the properties of

$$y_{ij} := 1^T Me_i \cdot (M_{jj} - 1) - 1^T Me_j \cdot M_{ji}$$

will be studied as functions of $\alpha_j$ and $P$ respectively in the next two lemmas.

**Lemma 3.7.** Given $\alpha \in [0, 1]^n$, let $A := \text{Diag}(\alpha)$ and $P$ be a row-stochastic matrix with $P_{ij} \neq 1$ for each $i \in [n]$. Suppose that the inverse $M := [I - (I - A)P]^{-1}$ exists. Let $j \in [n]$. Fix $P$ and all $\alpha_k$'s except $\alpha_j$. Then for any $i \in [n]$ with $i \neq j$,

$$y_{ij}(\alpha_j) := 1^T Me_i \cdot (M_{jj} - 1) - 1^T Me_j \cdot M_{ji}$$

is a strictly monotone function of $\alpha_j$ on $[0, 1]$ (either strictly increasing, strictly decreasing, or constantly zero). In addition, $y_{ij}(1) = 0$.

**Proof.** When $\alpha_j = 1$, note that the $j$-th row of the matrix $I - (I - A)P$ is equal to $e_j^T$. By considering the $j$-th row of the equation $[I - (I - A)P]M = I$, we have $M_{jj} = 1$ and $M_{jk} = 0$ for any $k \neq j$. Hence, $M_{jj} - 1 = M_{ji} = 0$ and so $y_{ij}(1) = 0$.

We will now show that $y_{ij}(\alpha_j)$ is a strictly monotone function of $\alpha_j$. Notice that $y_{ij}$ is a continuous function of $\alpha_j$ since $y_{ij}$ is a continuous function of $M$, $M$ is a continuous function of $\alpha_j$ (because of the continuity of matrix inversion), and a composition of continuous functions is continuous.

In addition, we know that $\frac{\partial B^{-1}}{\partial t} = -B^{-1} \frac{\partial B}{\partial t} B^{-1}$ for any invertible matrix $B$. Applying the above result with $B = I - (I - A)P$ and $t = \alpha_i$, we get

$$\frac{\partial M}{\partial \alpha_i} = -Me_i e_i^T PM.$$

Hence, when $\alpha_j \neq 1$, the partial derivative of $y_{ij}$ with respect to $\alpha_j$ is

$$\frac{\partial y_{ij}}{\partial \alpha_j} = \frac{\partial}{\partial \alpha_j} \left[ 1^T Me_i \cdot (e_j^T Me_j - 1) - 1^T Me_j \cdot e_j^T Me_i \right]$$

$$= 1^T \frac{\partial M}{\partial \alpha_j} e_i \cdot (e_j^T Me_j - 1) + 1^T Me_i \cdot e_j^T \frac{\partial M}{\partial \alpha_j} e_j + 1^T \frac{\partial M}{\partial \alpha_j} e_j \cdot e_j^T Me_i - 1^T Me_j \cdot e_j^T \frac{\partial M}{\partial \alpha_j} e_i$$

$$= -1^T Me_j e_i^T PM e_i \cdot (e_j^T Me_j - 1) - 1^T Me_i \cdot e_j^T Me_j e_j^T PM e_i + 1^T Me_j e_j^T PM e_j \cdot e_j^T Me_i + 1^T Me_j \cdot e_j^T Me_j e_j^T PM e_i$$
where \( M_{jj} \geq 1 > 0 \) by Lemma 3.4. Hence, the sign of \( \frac{\partial y_{ij}}{\partial \alpha_j} \) is always opposite to that of \( y_{ij} \) when \( \alpha_j \neq 1 \). Since \( y_{ij}(1) = 0 \) and \( y_{ij} \) is a continuous function of \( \alpha_j \), \( y_{ij} \) is strictly monotone on \( \alpha_j \in [0,1] \).

**Lemma 3.8.** Given \( \alpha \in [0,1]^n \), let \( A := \text{Diag}(\alpha) \) and \( P \) be a row-stochastic matrix with \( P_{ii} \neq 1 \) for each \( i \in [n] \). Suppose that the inverse \( M := [I - (I - P)A]^{-1} \) exists. For any distinct \( i, j \in [n] \), denote \( y_{ij} := 1^T M_{ei} \cdot (M_{jj} - 1) - 1^T M_{ej} \cdot M_{ji} \).

Then \( \sum_{k,l \in [n]} |\frac{\partial y_{ij}}{\partial P_{kl}}| \leq 4(1^T M 1)^3 \).

**Proof.** By the definition of the inverse of a matrix \( B \), we have \( BB^{-1} = I \). The partial derivative with respect to a variable \( t \) is: \( \frac{\partial B}{\partial t} B^{-1} + B \frac{\partial B^{-1}}{\partial t} = 0 \). Hence, we have \( \frac{\partial B^{-1}}{\partial t} = -B^{-1} \frac{\partial B}{\partial t} B^{-1} \). Applying the above result with \( B = I - (I - A)P \) and \( t = P_{kl} \) and denoting \( M = [I - (I - A)P]^{-1} \), we get

\[
\frac{\partial M}{\partial P_{kl}} = -M [-(I - A)e_k e_l^T] M = M(I - A)e_k e_l^T M = (1 - \alpha_k)M e_k e_l^T M.
\]

Hence, for any \( k,l \in \{0\} \cup [n] \),

\[
|\frac{\partial y_{ij}}{\partial P_{kl}}| = \left| 1^T (1 - \alpha_k)M e_k e_l^T M e_i (M_{jj} - 1) + 1^T M e_i e_j^T (1 - \alpha_k)M e_k e_l^T M e_j - 1^T (1 - \alpha_k)M e_k e_l^T M e_j M_{ji} - 1^T M e_j e_l^T (1 - \alpha_k)M e_k e_i^T M e_i \right|
\]

\[
= (1 - \alpha_k) \left| 1^T M e_k M_{li} (M_{jj} - 1) + 1^T M e_i M_{jk} M_{lj} - 1^T M e_k M_{ij} M_{ji} - 1^T M e_j M_{jk} M_{li} \right|
\]

\[
\leq \left| 1^T M e_k M_{li} (M_{jj} - 1) \right| + \left| 1^T M e_i M_{jk} M_{lj} \right| + \left| 1^T M e_k M_{ij} M_{ji} \right| + \left| 1^T M e_j M_{jk} M_{li} \right|
\]

\[
= 1^T M e_k M_{li} (M_{jj} - 1) + 1^T M e_i M_{jk} M_{lj} + 1^T M e_k M_{ij} M_{ji} + 1^T M e_j M_{jk} M_{li},
\]

where the last inequality holds because \( M \) is nonnegative by Lemma 3.4.
Thus, we obtain
\[
\sum_{k,l \in [n]} \left| \frac{\partial y_{ij}}{\partial P_{kl}} \right| \\
\leq \sum_{k=1}^{n} \sum_{i=1}^{n} 1^T Me_i M_{ij}(M_{jj} - 1) + 1^T Me_i M_{jk} M_{ij} + 1^T Me_i M_{ij} M_{ji} + 1^T Me_i M_{jk} M_{li} \\
= \sum_{k=1}^{n} 1^T Me_k 1^T Me_i (M_{jj} - 1) + 1^T Me_i M_{jk} 1^T Me_j + 1^T Me_k 1^T Me_j M_{ji} + 1^T Me_i M_{jk} 1^T Me_i \\
= 1^T M 11^T Me_i (M_{jj} - 1) + 1^T Me_i e_j^T M 11^T Me_j + 1^T M 11^T Me_j M_{ji} + 1^T Me_i e_j^T M 11^T Me_i \\
\leq 4(1^T M 1)^3
\]

where the last inequality is obtained by applying Lemma [3,4] again.

The interaction matrix chosen will eventually be a convex combination of the row-stochastic matrix that corresponds to the unweighted complete graph on the set of agents \(\{0\} \cup [n]\), as well as the row-stochastic matrix that corresponds to the union of a self-loop on agent 0 and the unweighted \(d\)-regular graph on \([n]\). Before that, the proof will first focus on the case of the unweighted complete graph. Define \(C := \frac{1}{n} (J - I) \in \mathbb{R}^{(n+1) \times (n+1)}\), where \(J := 11^T\) is the \((n + 1) \times (n + 1)\) matrix of ones. Then, \(C\) is the row-stochastic matrix which corresponds to the unweighted complete graph on \(\{0\} \cup [n]\). The following lemma shows that when \(C\) is the interaction matrix, the inverse \(M := [I - (I - A)C]^{-1}\) exists. Also, it provides an upper bound for the sum of entries of \(M\), i.e. \(1^T M 1\), and also some properties of \(y_{ij}\).

**Lemma 3.9.** Given a set of \(n + 1\) agents labelled \(\{0\} \cup [n]\) with \(\alpha_0 = 1\) and \(\alpha_k \in [0, 1]\) for all \(k \neq 0\).

Let \(C = \frac{1}{n} (J - I) \in \mathbb{R}^{(n+1) \times (n+1)}\) be the interaction matrix. Then \(M = [I - (I - A)C]^{-1}\), where \(A := \text{Diag}(\alpha)\), exists with \(1^T M 1 \leq n\). Also,

\[
y_{ij} := 1^T Me_i \cdot (M_{jj} - 1) - 1^T Me_j \cdot M_{ji} \leq 0
\]

for any distinct \(i, j \in [n]\) with equality holds if and only if \(\alpha_j = 1\). When \(\alpha_j = 0\), \(y_{ij} \leq -\frac{1}{n + 1}\) for any \(i \in [n]\) with \(i \neq j\).

**Proof.** By the Sherman-Morrison formula,

\[
C^{-1} = \left[\frac{1}{n} (J - I)\right]^{-1} = n \left[-I + 11^T\right]^{-1} = n \left[-I - \frac{(I) 11^T (I)}{1 + 11^T (I) 1}\right] \\
= n \left[-I - \frac{J}{1 - (n + 1)}\right] = J - nI,
\]

so we have

\[
M = [I - (I - A)C]^{-1} = [C^{-1} C - (I - A) C]^{-1} = C^{-1} (C^{-1} - I + A)^{-1} = (J - nI) [J + A - (n + 1) I]^{-1}.
\]

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Denote $D := A - (n + 1)I$. Then $D$ is a diagonal matrix with $D_{ii} = \alpha_i - n - 1 < 0$ for each $i$ and 

$$M = (J - nI)(J + D)^{-1}.$$ 

By the Sherman-Morrison formula again, we have 

$$(J + D)^{-1} = (D + 11^T)^{-1} = D^{-1} - \frac{D^{-1}11^T D^{-1}}{1 + 1^T D^{-1} 1} = D^{-1} - \frac{D^{-1}JD^{-1}}{1 + \sum_{k=0}^{n-1} \frac{1}{\alpha_k - n - 1}}.$$ 

Since $D^{-1}$ exists, $M^{-1}$ also exists.

Denote $w := \sum_{k=0}^{n} \frac{1}{\alpha_k - n - 1} < 0$. Then $(J + D)^{-1} = D^{-1} - \frac{D^{-1}JD^{-1}}{1 + w}$. We then have 

$$M = (J - nI) \left( D^{-1} - \frac{D^{-1}JD^{-1}}{1 + w} \right) = JD^{-1} - nD^{-1} - \frac{JD^{-1}JD^{-1}}{1 + w} + \frac{nD^{-1}JD^{-1}}{1 + w}.$$ 

Since $JD^{-1}J = 11^T D^{-1} 11^T = 1 \left( \sum_{k=0}^{n} \frac{1}{\alpha_k - n - 1} \right) 1^T = wJ$, 

$$M = JD^{-1} - nD^{-1} - \frac{w}{1 + w} JD^{-1} + \frac{n}{1 + w} D^{-1}JD^{-1}$$ 

and hence 

$$1^T M 1 = -\frac{1}{1 + w} 1^T JD^{-1} 1 - n 1^T D^{-1} 1 + \frac{n}{1 + w} 1^T D^{-1}JD^{-1} 1$$ 

$$= \frac{1}{1 + w} 1^T 11^T D^{-1} 1 - n 1^T D^{-1} 1 + \frac{n}{1 + w} (1^T D^{-1} 1)^2$$ 

$$= (n + 1)w - nw + \frac{nw^2}{1 + w} = \frac{w}{1 + w}.$$ 

Note that 

$$1 + w = 1 + \sum_{k=0}^{n} \frac{1}{\alpha_k - n - 1} = \sum_{k=0}^{n} \left( \frac{1}{n+1} + \frac{1}{\alpha_k - n - 1} \right),$$ 

and 

$$\frac{1}{n+1} + \frac{1}{\alpha_k - n - 1} = \frac{\alpha_k - n - 1 + n + 1}{(n+1)(\alpha_k - n - 1)} = \frac{\alpha_k}{(n+1)(\alpha_k - n - 1)} < 0$$ 

for any $k \in \{0\} \cup [n]$ with strict inequality holds for $k = 0$ since $\alpha_0 = 1$.

Therefore, $1 + w < 0$ and thus 

$$1^T M 1 = \frac{w}{1 + w} \leq \frac{\sum_{k=0}^{n} \frac{1}{\alpha_k - n - 1}}{1 + \sum_{k=0}^{n} \frac{1}{\alpha_k - n - 1}} = 1 + \frac{n + 1}{n} = n.$$
Also, from Equation 1 we have, for any $k \in [n]$,
\[
Me_k = \frac{1}{1+w}JD^{-1}e_k - nD^{-1}e_k + \frac{n}{1+w}D^{-1}JD^{-1}e_k \\
= \frac{1}{\alpha_k - n - 1} \left( \frac{1}{1+w} - ne_k + \frac{n}{1+w}D^{-1} \right).
\]

Then for any distinct $i, j \in [n]$,
\[
y_{ij} = 1^T Me_i \cdot (M_{jj} - 1) - 1^T Me_j \cdot M_{ji} \\
= 1^T Me_i \cdot (e_j^T Me_j - 1) - 1^T Me_j \cdot e_j^T Me_i \\
= \frac{1}{\alpha_i - n - 1} \left( \frac{n+1}{1+w} - n + \frac{nw}{1+w} \right) \cdot \left[ \frac{1}{\alpha_j - n - 1} \left( \frac{1}{1+w} - n + \frac{n}{1+w}e_{jj} - \frac{1}{\alpha_j - n - 1} \right) - 1 \right] - \\
\frac{1}{\alpha_j - n - 1} \left( \frac{n+1}{1+w} - n + \frac{nw}{1+w} \right) \cdot \left[ \frac{1}{\alpha_i - n - 1} \left( \frac{1}{1+w} + \frac{n}{1+w}e_{ii} - \frac{1}{\alpha_i - n - 1} \right) \right] \\
= \frac{-n - (\alpha_j - n - 1)}{(\alpha_i - n - 1)(\alpha_j - n - 1)} \left( \frac{n+1}{1+w} - n + \frac{nw}{1+w} \right) \\
= \frac{1 - \alpha_j}{(\alpha_i - n - 1)(\alpha_j - n - 1)(1+w)}. \tag{2}
\]

Since the denominator is negative, $y_{ij} \leq 0$ with equality holds if and only if $\alpha_j = 1$.

Finally, when $\alpha_j = 0$, Equation 2 gives
\[
y_{ij} = \frac{1 - \alpha_j}{(\alpha_i - n - 1)(\alpha_j - n - 1)(1+w)} \\
= \frac{1}{\alpha_i - n - 1} \left( \frac{1}{\alpha_j - n - 1} - \frac{1}{0} \right) \\
\leq \frac{1}{(-n-1)(-n-1)(1 + \frac{n}{1-n-1} + \frac{1}{-n-1})} \\
= \frac{1}{(n+1)^2 \left( -\frac{1}{n+1} \right)} \\
= -\frac{1}{n+1}
\]
as desired. \[\square\]

Let $R \in \mathbb{R}^{n \times n}$ be a row-stochastic matrix that corresponds to a graph with $n$ vertices. Define $G := \begin{bmatrix} 1 & 0^T \\ 0 & R \end{bmatrix}$. Then $G$ forms an interaction matrix on the set of agents $\{0\} \cup [n]$, with agent 0 being isolated from any other agent. We will introduce a perturbation to the interaction matrix $C$ by the matrix $G$, i.e. $C$ is perturbed to $P := (1 - \delta)C + \delta G$ for some $0 < \delta < 1$. Then, as we will show in the following lemma, the matrix inverse $[I - (I-A)P]^{-1}$ also exists.
Lemma 3.10. Given a set of \( n + 1 \) agents labelled \( \{0\} \cup [n] \) with \( \alpha_0 = 1 \) and \( \alpha_k \in [0, 1] \) for all other \( k \)'s. Let \( C = \frac{1}{n}(I - I) \in \mathbb{R}^{(n+1) \times (n+1)} \) be an interaction matrix. Let also \( R \in \mathbb{R}^{n \times n} \) be a row-stochastic matrix that corresponds to a graph with \( n \) vertices. Define \( G := \begin{bmatrix} 1 & 0^T \\ 0 & R \end{bmatrix} \) and \( P = (1 - \delta)C + \delta G \), where \( 0 \leq \delta < 1 \). Then, \( M := [I - (I - A)P]^{-1} \), where \( A := \text{Diag}(\alpha) \), exists.

Proof. The case when \( \delta = 0 \) is proven in Lemma 3.9. Consider \( 0 < \delta < 1 \). The Woodbury matrix identity suggests that the inverse of \( I - (I - A)P \), if exists, is given by

\[
[I - (I - A)P]^{-1} = [I + (I - A)(I - A)]^{-1}P^{-1} \\
= I^{-1} - I^{-1}(I - A)[(I - A) + PI^{-1}(I - A)]^{-1}PI^{-1} \\
= I + (I - A)[I - P(I - A)]^{-1}P.
\]

To show that \( I - (I - A)P \) is invertible, we show that \( [I - P(I - A)]^{-1} \) exists.

By Neumann Series Theorem, it suffices to show that the induced \( \infty \)-norm of \( Q := P(I - A) \) is strictly less than 1, i.e. \( \|Q\|_{\infty} = \max_{0 \leq i \leq n} \sum_{j=0}^{n} |Q_{ij}| < 1 \). But since \( \alpha_0 = 1 \),

\[
\sum_{j=0}^{n} |Q_{ij}| = \sum_{j=0}^{n} P_{ij}(1 - \alpha_j) = \sum_{j=1}^{n} P_{ij}(1 - \alpha_j) \leq \sum_{j=1}^{n} P_{ij} = \begin{cases} 1 - \delta & \text{if } i = 0 \\ 1 - \frac{1 - \delta}{n} & \text{if } i \neq 0 \end{cases}.
\]

Since \( 0 < \delta < 1 \), \( \|Q\|_{\infty} < 1 \). Hence the result. \( \square \)

The next lemma, first given in Alfa et al. [11], gives two-sided bounds to the inverse of a perturbed nonsingular diagonally dominant \( M \)-matrix. Since \( X := I - (I - A)P \) is an \( M \)-matrix, the lemma will be used to bound the sum of entries of the matrix inverse \( M := X^{-1} \), i.e. \( 1^T M 1 \), when \( P \) is obtained by perturbing \( C \) slightly.

Lemma 3.11 (Entrywise bounds for diagonally dominant \( M \)-matrix inverse [11]). Let \( 0 \leq \varepsilon < 1 \). If \( A = [a_{ij}] \) and \( \bar{A} = [\bar{a}_{ij}] \) are two \( n \times n \) nonsingular diagonally dominant \( M \)-matrices with \( |a_{ij} - \bar{a}_{ij}| \leq \varepsilon |a_{ij}| \) for \( i \neq j \) and \( |A1 - \bar{A}1| \leq \varepsilon |A1| \), then

\[
\frac{(1 - \varepsilon)^n}{(1 + \varepsilon)^{n-1}} A^{-1} \leq \bar{A}^{-1} \leq \frac{(1 + \varepsilon)^n}{(1 - \varepsilon)^{n-1}} A^{-1}.
\]

Consider the interaction matrix \( G := \begin{bmatrix} 1 & 0^T \\ 0 & R \end{bmatrix} \), where \( R \in \mathbb{R}^{n \times n} \) corresponds to an unweighted \( d \)-regular graph on the set of agents \( [n] \). Now assume that the interaction matrix \( P \) is obtained by perturbing the interaction matrix \( C \) by \( G \). The next lemma will give an upper bound for the sum of entries of the matrix inverse \( M = [I - (I - A)P]^{-1} \), i.e. \( 1^T M 1 \).
Lemma 3.12. Given a set of $n+1$ agents labelled \{0\} $\cup$ [n] with $\alpha_0 = 1$ and $\alpha_k \in [0,1]$ for all other $k$'s. Let $C = \frac{1}{n} \frac{1}{(J-I) \in R^{(n+1) \times (n+1)}}$ be an interaction matrix. Let also $R \in R^{n \times n}$ be a row-stochastic matrix which corresponds to an unweighted $d$-regular graph with $n$ vertices. Define 

\[ G := \begin{bmatrix} 1 & 0^T \\ 0 & R \end{bmatrix} \text{ and } P(\delta) = (1-\delta)C + \delta G \text{ for any } 0 \leq \delta < \frac{d}{n}. \]

Let $X(\delta) = I - (I-A)P(\delta)$ where $A := \text{Diag}(\alpha)$. Then, $M(\delta) := X^{-1}(\delta)$ exists. Moreover,

\[ 1^T M(\delta) 1 \leq \frac{n(1+\varepsilon)^n}{(1-\varepsilon)^{n-1}} \text{ where } \varepsilon = \frac{\delta n}{d}. \]

Proof. By Lemma 3.4, $X(\delta)$ is a diagonally dominant $M$-matrix for any $0 \leq \delta < \frac{d}{n}$. Also, the existence of $M(\delta)$ for $0 \leq \delta < \frac{d}{n}$ is shown in Lemma 3.10.

For any distinct $i,j \neq 0$, since $G_{ij} = 0$ or $G_{ij} = \frac{1}{d}$, we have

\[
|X_{ij}(0) - X_{ij}(\delta)| = \left| \frac{1-\alpha_i}{n} + (1-\alpha_i)(1-\delta) \right| = \frac{\delta(1-\alpha_i)}{n} = \delta |X_{ij}(0)| < \varepsilon |X_{ij}(0)|,
\]

\[
|X_{0j}(0) - X_{0j}(\delta)| = |0-0| = 0 < \varepsilon |X_{0j}(0)|,
\]

\[
|X_{ij}(0) - X_{ij}(\delta)| = \left| \frac{1-\alpha_i}{n} + (1-\alpha_i) \left( \frac{1-\delta}{n} + \delta G_{ij} \right) \right| = \frac{\delta(1-\alpha_i)}{n} \frac{\delta(1-\alpha_i)}{d} = \frac{\delta n}{d} \frac{1-\alpha_i}{n} = \varepsilon |X_{ij}(0)|.
\]

In addition, as $C$ and $P$ are row-stochastic matrices, we have $P1 = C1 = 1$ and hence

\[ |X(0)1 - X(\delta)1| = |X(0) - X(\delta)|1| = |\delta(I-A)(P-C)1| = 0 \leq \varepsilon |X(0)1|. \]

Therefore, we have $M(\delta) \leq \frac{(1+\varepsilon)^n}{(1-\varepsilon)^{n-1}} M(0)$ by Lemma 3.11. Then, as $1^T M(0)1 \leq n$ by Lemma 3.9, we have

\[ 1^T M(\delta) 1 \leq \frac{(1+\varepsilon)^n}{(1-\varepsilon)^{n-1}} 1^T M(0)1 \leq \frac{n(1+\varepsilon)^n}{(1-\varepsilon)^{n-1}} \]

as desired. \qed

With the previous preparation, the next lemma proposes an exact universal perturbation parameter $\delta$ that guarantees the negativity of $y_{ij}$ when $\alpha_j \in [0,1)$ for any distinct $i,j \in [n]$.

Lemma 3.13. Given a set of $n+1$ agents labelled \{0\} $\cup$ [n] with $\alpha_0 = 1$ and $\alpha_k \in [0,1]$ for all other $k$'s. Let $C = \frac{1}{n} \frac{1}{(J-I) \in R^{(n+1) \times (n+1)}}$ be an interaction matrix. Let also $R \in R^{n \times n}$ be a row-stochastic matrix which corresponds to an unweighted $d$-regular graph with $n$ vertices. Define
Proof. Note that 

\[ \text{Let } \delta \text{ so } \delta > 0. \]

Then, \( M(t) := [I - (I - A)P(t)]^{-1} \), where \( A := \text{Diag}(\alpha) \), exists. Define, for any distinct \( i, j \in [n] \) and \( 0 \leq t < \frac{d}{n} \),

\[ y_{ij}(t) := 1^T M(t) e_i \cdot (M_{jj}(t) - 1) - 1^T M(t) e_j \cdot M_{ji}(t). \]

Let \( \delta = \frac{d^3(2d - 1)^{3n-3}}{(n + 1)^6(2d + 1)^{3n}} \). Then \( 0 < \delta < \frac{d}{n} \). For any distinct \( i, j \in [n] \), if \( \alpha_j \in [0, 1) \), then \( y_{ij}(\delta) < 0 \).

Proof. Note that

\[
P_{ij}(t) = \begin{cases} 
1-t^2 & \text{if } i = j = 0 \\
\frac{1-t^2}{n} & \text{if } i \neq j \text{ and either } i = 0 \text{ or } j = 0 \\
0 & \text{if } i = j \neq 0 \\
\frac{1-t^2}{n} & \text{if } i \neq j \text{ and both } i, j \neq 0 
\end{cases}.
\]

so

\[
P'_{ij}(t) = \begin{cases} 
1 & \text{if } i = j = 0 \\
\frac{1}{n} & \text{if } i \neq j \text{ and either } i = 0 \text{ or } j = 0 \\
0 & \text{if } i = j \neq 0 \\
\frac{1}{n} & \text{if } i \neq j \text{ and both } i, j \neq 0 
\end{cases}.
\]

Therefore, treating \( P \) as a \((n + 1)^2\)-dimensional vector, we have \( \|P'(t)\|_1 = \sum_{i=0}^{n} \sum_{j=0}^{n} |P'_{ij}(t)| \leq (n + 1)^2 \).

Hence,

\[
|y_{ij}(\delta) - y_{ij}(0)| = \left| \int_0^\delta \nabla y_{ij}(t) \cdot P'(t) \, dt \right| \quad \text{(Fundamental theorem for line integrals)}
\]

\[
\leq \int_0^\delta |\nabla y_{ij}(t) \cdot P'(t)| \, dt 
\]

\[
\leq \int_0^\delta \|\nabla y_{ij}(t)\|_2 \cdot \|P'(t)\|_2 \, dt \quad \text{(Cauchy-Schwarz Inequality)}
\]

\[
\leq \int_0^\delta \|\nabla y_{ij}(t)\|_1 \cdot \|P'(t)\|_1 \, dt 
\]

\[
\leq \int_0^\delta \left[ \sum_{k=0}^{n} \sum_{l=0}^{n} \left| \frac{\partial y_{ij}(t)}{\partial P_{kl}} \right| \right] \cdot (n + 1)^2 \, dt 
\]

\[
\leq \int_0^\delta 4(1^T M(t) 1)^3 \cdot (n + 1)^2 \, dt \quad \text{(Lemma 3.8)}
\]

\[
\leq \int_0^\delta 4 \left[ \frac{n(1 + \varepsilon)^n}{(1 - \varepsilon)^{n-1}} \right]^3 \cdot (n + 1)^2 \, dt \quad \text{(Lemma 3.12)}
\]

\[
\leq \frac{4\delta(n + 1)^{3} (1 + \varepsilon)^{3n}}{(1 - \varepsilon)^{3n-3}}
\]

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where \( \epsilon = \frac{\delta n}{d} \). Since \( \delta = \frac{d^3(2d-1)^{3n-3}}{(n+1)^6(2d+1)^{3n}} < \frac{1}{2n} \), \( \epsilon = \frac{\delta n}{d} < \frac{1}{2d} \). Hence, we have

\[
|y_{ij}(\delta) - y_{ij}(0)| < \frac{4\delta(n+1)^2(1 + \frac{1}{2d})^{3n}}{(1 - \frac{1}{2d})^{3n-3}} = \frac{\delta(n+1)^2(2d+1)^{3n}}{2d^3(2d-1)^{3n-3}}
\]

Also, since \( \delta = \frac{d^3(2d-1)^{3n-3}}{(n+1)^6(2d+1)^{3n}} \), we have

\[
|y_{ij}(\delta) - y_{ij}(0)| < \frac{(n+1)^5(2d+1)^{3n}}{2d^3(2d-1)^{3n-3}} \cdot \frac{d^3(2d-1)^{3n-3}}{(n+1)^6(2d+1)^{3n}} = \frac{1}{2(n+1)}
\]

for any \( \alpha \) with \( \alpha_0 = 1 \) and \( \alpha_k \in [0, 1] \) for any other \( k \)'s. In particular, this is true when \( \alpha_j \) is fixed at 0.

From Lemma 3.9, \( y_{ij}(0) \leq -\frac{1}{n+1} \) when \( \alpha_j = 0 \). Hence, \( y_{ij}(\delta) < -\frac{1}{2(n+1)} < 0 \) when \( \alpha_j = 0 \).

Then by Lemma 3.7, \( y_{ij}(\delta) < 0 \) for any \( \alpha_j \in [0, 1] \).

Below is the last lemma before the proof of the NP-hardness of the L1-budgeted problem. It suggests that using the proposed perturbation parameter \( \delta \) in Lemma 3.13, if there is \( k \) units of budget available where \( k \in \mathbb{N} \), then there exists an optimal assignment such that exactly \( k \) agents receive a unit of budget.

**Lemma 3.14.** Given a set of \( n+1 \) agents labelled \( \{0\} \cup [n] \) with \( \alpha_0 = 1 \) and \( \alpha_k \in [0, 1] \) for all other \( k \)'s. Set \( s_i = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \). Let \( C = \frac{1}{n} (J - I) \in \mathbb{R}^{(n+1) \times (n+1)} \) be an interaction matrix. Let also \( R \in \mathbb{R}^{n \times n} \) be a row-stochastic matrix which corresponds to an unweighted \( d \)-regular graph with \( n \) vertices. Define \( G := \begin{bmatrix} 1 & 0^T \\ 0 & R \end{bmatrix} \) and \( P = (1 - \delta)C + \delta G \), where \( \delta = \frac{d^3(2d-1)^{3n-3}}{(n+1)^6(2d+1)^{3n}} \). Then \( M := (I - (I - A)P)^{-1} \), where \( A := \text{Diag}(\alpha) \), exists.

For any distinct \( i, j \in [n] \), fix all \( \alpha_k \)'s except \( \alpha_i \) and \( \alpha_j \). Let \( 0 < b < 2 \). On the line segment \( S := \{ \alpha \in [0, 1]^{n+1} : \alpha_i + \alpha_j = b \} \), we have

\[
\min_{\alpha \in S} f(\alpha) = \min_{\alpha \in \partial S} f(\alpha),
\]

where \( \partial S \) denotes the boundary of \( S \) and \( f(\alpha) := 1^T M \alpha \) is the sum of equilibrium opinions.

**Proof.** We will show that every local stationary point of \( f \) on the interior of \( S \) is a local maximum point on \( S \). Suppose \( \alpha \in S \) is a local stationary point of \( f \) with \( \alpha_i, \alpha_j \notin [0, 1] \). By Lemma 3.13, \( y_{ij} < 0 \) and \( y_{ji} < 0 \). Moreover, since agents \( i \) and \( j \) are both connected to agent 0 whose innate opinion is 1, and both \( \alpha_i \) and \( \alpha_j \) are nonzero, we have \( z_i(\alpha) > 0 \) and \( z_j(\alpha) > 0 \). Hence, according to Lemma 3.6, the second directional derivative of \( f \) at \( \alpha \) in the direction \( e_i - e_j \) is negative, so \( \alpha \) is indeed a local maximum point on \( S \).
Lastly, the proof of the NP-hardness of the L1-budgeted problem is presented in the following theorem.

**Theorem 3.15.** The L1-budgeted Opinion Susceptibility Problem is NP-hard.

**Proof.** We give a reduction from the vertex cover problem for regular graphs. Given a $d$-regular graph $H = ([n], E)$ and an integer $B$, the vertex cover problem asks whether there exists a set $S$ of vertices with size at most $B$ such that $S$ is a vertex cover, i.e., every edge in $E$ is incident to at least one node in $S$.

**Reduction Construction.** Construct an instance of the decision version of the L1-budgeted Opinion Susceptibility Problem with $n+1$ agents. Label them $\{0\} \cup [n]$. The innate opinions, initial resistance parameters and the bounds of resistance parameters are

$$s_i = \alpha_i^{(0)} = l_i = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases} \quad \text{and} \quad u_i = 1 \text{ for all } i.$$

Define $R \in \mathbb{R}^{n \times n}$ by $R_{ij} := \begin{cases} \frac{1}{d} & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$ for any $i, j \in [n]$ and $G := \begin{bmatrix} 1 & 0^T \\ 0 & R \end{bmatrix}$, which is a row-stochastic matrix of size $(n+1) \times (n+1)$. The interaction matrix of the instance is $P = (1 - \delta)C + \delta G$, where $C = \frac{1}{n}(J - I)$ is the row-stochastic matrix that corresponds to an unweighted complete graph on $\{0\} \cup [n]$ and $\delta = \frac{d^3(2d-1)^{3n-3}}{(n+1)^6(2d+1)^n}$.

Finally, we set the budget $b = B$ and the threshold $\theta = 1 + \frac{(1 - \delta)(n-b)}{n - (1 - \delta)(n-b-1)}$. To complete the reduction proof, we show that there exists a vertex cover of size $k$ in $H$ if and only if there exists some $\alpha \in \prod_{i=0}^n [l_i, u_i]$ such that $\|\alpha - \alpha^{(0)}\|_1 \leq b$ and $f(\alpha) \leq \theta$.

**Forward Direction.** Suppose in $H$, there is some vertex cover $T \subset [n]$ with size $b$. We assign a unit budget to each agent in $T$ so that $\alpha_i = 1$ for each $i \in T$, and we will show that this assignment gives $f(\alpha) \leq \theta$. We shall analyse the equilibrium opinion of each agent. Note that agent 0 has equilibrium opinion 1 while agent $i$ has equilibrium opinion 0 for any $i \in T$ since $\alpha_i = 1$ for any $i \in \{0\} \cup T$. To analyse the equilibrium opinion of the other agents, we rearrange the order of the agents such that the first agent is agent 0 and the last $n-b$ agents do not belong to $\{0\} \cup T$.

Denote the $n-b$ agents by $j_1, \cdots, j_{n-b}$. Also, define $G(i, j) = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$ for any $i, j \in [n]$ and $z_j = (z_{j_1}, \cdots, z_{j_{n-b}})^T$. Then by the equation $z = As + (I - A)Pz$ about the equilibrium opinion vector $z$, we have

$$\begin{bmatrix} 1 \\ 0 \\ z_j \end{bmatrix} = \begin{bmatrix} 1 \\ I \\ O \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ O \\ I \end{bmatrix} [(1 - \delta)C + \delta G] \begin{bmatrix} 1 \\ 0 \\ z_j \end{bmatrix}$$
Let the budget for any \( i \in \{0, \ldots, n \} \) be any arbitrary subset of \( \mathbb{Z}^n \) such that the first agent is agent 0 and the last \( n \) agents such that the first agent is agent 0 and the last \( n \) agents. Therefore, adding the last \( n - b \) coordinates, we have

\[
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix} + \begin{bmatrix}
O & 1 \\
1 & \iddots \\
\vdots & \ddots & 1 \\
0 & \cdots & 1
\end{bmatrix} \begin{bmatrix}
1 - \frac{n}{1 - \delta} \\
1 + \sum_{i=1}^{n-b} z_{j_i} - z_{j_1} \\
\vdots \\
1 + \sum_{i=1}^{n-b} z_{j_i} - z_{j_{n-b}}
\end{bmatrix} = \begin{bmatrix}
* \\
* \\
\vdots \\
* \\
\sum_{i=1}^{n-b} G(j_1, j_i) z_{j_i}
\end{bmatrix} + \frac{\delta}{d} \begin{bmatrix}
1 \\
1 + \sum_{i=1}^{n-b} z_{j_i} - z_{j_{n-b}} \\
\vdots \\
\sum_{i=1}^{n-b} G(j_{n-b}, j_i) z_{j_i}
\end{bmatrix}
\]

(3)

Note that after the re-ordering, the \((n - b) \times (n - b)\) submatrix of \( G \) obtained by deleting the first \( b + 1 \) rows and the first \( b + 1 \) columns is the zero matrix, or otherwise some edge \((j_p, j_q) \in E\) is not covered by \( T \). Hence, \( G(j_m, j_1) = 0 \) for any \( l, m \in \{1, \ldots, n - b\} \).

This gives

\[
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix} + \begin{bmatrix}
0 & 1 \\
1 & \iddots \\
\vdots & \ddots & 1 \\
0 & \cdots & 1
\end{bmatrix} \begin{bmatrix}
1 - \frac{n}{1 - \delta} \\
1 + \sum_{i=1}^{n-b} z_{j_i} - z_{j_1} \\
\vdots \\
1 + \sum_{i=1}^{n-b} z_{j_i} - z_{j_{n-b}}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
\sum_{i=1}^{n-b} G(j_1, j_i) z_{j_i}
\end{bmatrix} + \frac{\delta}{d} \begin{bmatrix}
0 \\
\sum_{i=1}^{n-b} G(j_{n-b}, j_i) z_{j_i}
\end{bmatrix}
\]

Adding the last \( n - b \) coordinates, we have

\[
\sum_{i=1}^{n-b} z_{j_i} = \frac{1 - \frac{n}{1 - \delta}}{\frac{n}{1 - \delta} - \frac{n - b}{1 - \delta}} \left( n - b + (n - b - 1) \sum_{i=1}^{n-b} z_{j_i} \right),
\]

or

\[
\sum_{i=1}^{n-b} z_{j_i} = \frac{(1 - \delta)(n - b)}{n - (1 - \delta)(n - b - 1)}.
\]

(4)

Therefore, \( f(\alpha) = 1 + \sum_{i=1}^{n-b} z_{j_i} = \theta \).

**Backward Direction.** Suppose in \( H \), there is no vertex cover of size \( b \). By Lemma 3.14, allocating the \( b \) units of budget to more than \( b \) agents cannot lead to the minimum sum of equilibrium opinions. Let \( T \) be any arbitrary subset of \( [n] \) of size \( b \). We assign a unit budget to each agent in \( T \). Note that agent 0 has equilibrium opinion 1 while agent \( i \) has equilibrium opinion 0 for any \( i \in T \) since \( \alpha_i = 1 \) for any \( i \in \{0\} \cup T \). To analyse the equilibrium opinion of the other agents, we rearrange the order of the agents such that the first agent is agent 0 and the last \( n - b \) agents do not belong to \( \{0\} \cup T \). If we denote the \( n - b \) agents \( j_1, \ldots, j_{n-b} \), then the \((n - b) \times (n - b)\) submatrix of \( G \) obtained by deleting the first \( b + 1 \) rows and the first \( b + 1 \) columns is nonzero, or otherwise \( T \) forms a vertex cover. Hence, we have \( G(j_l, j_m) = 1 \) for some \( l, m \in \{1, \ldots, n - b\} \). Since agent \( j_l \) is connected to
agent 0 whose equilibrium opinion is 1 and \( \alpha_{jm} = 0, z_{jm} > 0 \). Adding the last \( n-b \) coordinates of the equilibrium opinion vector in Equation 3, we have

\[
\sum_{i=1}^{n-b} z_{ji} \geq \frac{1 - \delta}{n} \left[ n - b + (n - b - 1) \sum_{i=1}^{n-b} z_{ji} \right] + \frac{\delta z_{jm}}{d},
\]

or

\[
\sum_{i=1}^{n-b} z_{ji} \geq \frac{(1 - \delta)(n - b)}{n - (1 - \delta)(n - b - 1)} + \epsilon,
\]

where \( \epsilon = \frac{\delta n z_{jm} / d}{n - (1 - \delta)(n - b - 1)} > 0 \). Therefore, \( f(\alpha) \geq 1 + \frac{(1 - \delta)(n - b)}{n - (1 - \delta)(n - b - 1)} + \epsilon > \theta \).

It remains to establish that \( \epsilon \) can be represented using a polynomial number of bits by showing that this can be done for \( \delta \) and \( z_{jm} \). Indeed, since \( \delta = \frac{d^3(2d - 1)^{3n-3}}{(n+1)^6(2d+1)^{3n}} < 1 \), the number of bits required to represent \( \delta \) is

\[
|\log \delta| \leq 6 \log(n+1) + 3n \log(2d + 1) \leq 6 \log(n+1) + 3n \log(2n + 1).
\]

Moreover, we know that if a unit of budget were allocated to every agent \( i \in [n] \) except \( j_m \), the equilibrium opinion of \( j_m \) would not be higher. Since any graph with \( n \) vertices must contain a vertex cover of size \( n - 1 \), the equilibrium opinion of \( j_m \) would become \( 1 - \frac{\delta}{n} \) according to Equation 4. Therefore, \( z_{jm} \geq 1 - \frac{\delta}{n} \), which can also be represented using a polynomial number of bits.

The proof is completed.
4 Properties of the Objective Function

In view of the NP-hardness result in Section 3, a gradient-based algorithm will be devised to approximate the optimal solution of any $L_1$-budgeted problem. Chan et al. [2] and Abebe et al. [12] proved some properties of the objective function to support their algorithms that solve the unbudgeted variant and the $L_0$-budgeted variant. Among the proven properties, the marginal monotonicity of the objective function, the approximation of the objective function, and the gradient approximation will be essential in the devised algorithm.

4.1 Marginal monotonicity

As suggested in [2], the objective function $f$ is generally non-convex, but it is marginally monotone. The following lemma, proven in [2], shows that the sign of the partial derivatives of $f$ with respect to $\alpha_i$ is independent of $\alpha_i$. The lemma also implies that fixing all except one coordinate of $\alpha$, $f$ is either strictly increasing, strictly decreasing, or constant.

**Lemma 4.1** (Sign of Partial Derivatives Independent of Coordinate Value). For any $\alpha_i \in (0, 1)^n$ and any $i \in [n]$, the sign of $\frac{\partial f(\alpha)}{\partial \alpha_i}$ is independent of $\alpha_i$.

Lemma 4.1 implies that in the unbudgeted variant, it suffices to consider the extreme points $\alpha_i \in \{l_i, u_i\}$ for each $i \in [n]$. Regarding the $L_1$-budgeted variant, we shall see later that this lemma will be useful for locating the optimal solution in Section 5.2.

4.2 Approximation of the Objective Function

The objective function of the Opinion Susceptibility Problem is a function of the resistance vector $\alpha$ given by $f(\alpha) = 1^T z(\alpha) = 1^T [I - (I - A)P]^{-1} As$, where $A = \text{Diag}(\alpha)$. Since computing matrix inverse is an expensive operation, Chan et al. [2] approximated $z(\alpha)$ using the recurrence $z(0) \in [0, 1]^n$ and $z(t+1) := As + (I - A)P z(t)$. The following lemma gives an upper bound on the additive error for each coordinate.

**Lemma 4.2** (Approximation Error of the Equilibrium Opinion Vector). Suppose for some $\varepsilon > 0$, for all $i \in [n]$, $\alpha_i \geq \varepsilon$. Then, for every $t \geq 0$, $\|z(\alpha) - z(t)\|_{\infty} \leq \frac{(1 - \varepsilon)^t}{\varepsilon}$.

Denote $\varepsilon_\alpha := \min_{i \in [n]} \alpha_i$. Lemma 4.2 suggests that if the recurrence described is used to approximate $f(\alpha)$ with the maximum absolute error $\delta$, then $\left\lceil \frac{\ln \delta + \ln \varepsilon_\alpha - \ln n}{\ln (1 - \varepsilon_\alpha)} \right\rceil$ iterations are sufficient. Algorithm [1] provides the details about the iterative process of evaluating the objective function at a given resistance vector $\alpha$. 

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Algorithm 1: Objective Approximation

**Input:** Resistance vector \( \alpha \); innate opinions \( s \in [0, 1]^n \); interaction matrix \( P \); approximation parameter \( \delta \).

**Output:** Objective value \( f^* \).

1. Set \( \varepsilon_\alpha \leftarrow \min_{i \in [n]} \alpha_i \), \( T \leftarrow \left\lceil \frac{\ln \delta + \ln \varepsilon_\alpha - \ln n}{\ln (1 - \varepsilon_\alpha)} \right\rceil + 1 \), \( z \leftarrow 1 \), and \( t \leftarrow 0 \).

2. while \( t < T \) do

3. \( z \leftarrow As + (I - A)Pz \), where \( A = \text{Diag}(\alpha) \)

4. \( t \leftarrow t + 1 \)

5. return Objective value \( f^* = \sum_{i \in [n]} z_i \).

4.3 Approximation of the Gradient Function

Chan et al. [2] proved that the partial derivatives of \( f \) with respect to \( \alpha_i \) is equal to \( \frac{\partial f(\alpha)}{\partial \alpha_i} = \frac{s_i - z_i(\alpha)}{1 - \alpha_i} \cdot e_i^T [I - P^T (I - A)]^{-1} 1 \). Again, to avoid the computation of matrix inverse, Abebe et al. [12] approximated \( r(\alpha) := [I - P^T (I - A)]^{-1} 1 \) using the recurrence \( r(0) \in [0, 1]^n \) and \( r(t+1) := 1 + P^T (I - A) r(t) \). The following lemma gives an upper bound on the error sum of the coordinates of \( r \).

**Lemma 4.3** (Approximation Error of \( r \)). Suppose for some \( \varepsilon > 0 \), for all \( i \in [n] \), \( \alpha_i \geq \varepsilon \). Then, for every \( t \geq 0 \), \( \| r(\alpha) - r(t) \|_1 \leq n \cdot \frac{(1 - \varepsilon)^t + 1}{\varepsilon} \).

Denote \( \varepsilon_\alpha := \min_{i \in [n]} \alpha_i \). Lemma 4.3 suggests that if the recurrence described is used to approximate \( r(\alpha) \) with the maximum absolute error \( \delta \), then \( \left\lceil \frac{\ln \delta + \ln \varepsilon_\alpha - \ln n}{\ln (1 - \varepsilon_\alpha)} \right\rceil \) iterations are sufficient.

Algorithm 2 provides the details about the iterative process of evaluating the gradient function of \( f \) at a given resistance vector \( \alpha \).

Algorithm 2: Gradient Approximation

**Input:** Resistance vector \( \alpha \); innate opinions \( s \in [0, 1]^n \); interaction matrix \( P \); approximation parameter \( \delta \).

**Output:** Gradient vector \( g \in \mathbb{R}^n \).

1. Set \( \varepsilon_\alpha \leftarrow \min_{i \in [n]} \alpha_i \), \( T \leftarrow \left\lceil \frac{\ln \delta + \ln \varepsilon_\alpha - \ln n}{\ln (1 - \varepsilon_\alpha)} \right\rceil + 1 \), \( z \leftarrow 1 \), \( r \leftarrow 1 \), \( r \leftarrow 1 \), and \( t \leftarrow 0 \).

2. while \( t < T \) do

3. \( z \leftarrow As + (I - A)Pz \), where \( A = \text{Diag}(\alpha) \)

4. \( r \leftarrow 1 + P^T (I - A) r \)

5. \( t \leftarrow t + 1 \)

6. return Gradient vector \( g = \text{Diag}(s - z) \text{Diag}(1 - \alpha)^{-1} r \).
5 Feasible Region and Search Space

This section will analyse the location of the L1-budgeted optimal solution. Specifically, the analysis will reduce the search space to the union of some faces and some extreme points of the feasible region.

5.1 Feasible region

Two types of linear constraints determine the feasible region of the L1-budgeted variant, namely (1) the upper and the lower bounds of the resistance parameters and (2) the budget constraint. The feasible region of the L1-budgeted variant is a convex subset of that of the unbudgeted variant with the same set of parameters $s, P, l, \text{ and } u$. This is because the feasible region of the L1-budgeted variant is the intersection between the unbudgeted feasible region and the L1-norm ball centred at $\alpha^{(0)}$ with radius $b$, both of which are convex. Figure 1a and Figure 1b show the possible configuration of the feasible region of the L1-budgeted variant when $n = 2$ and when $n = 3$ respectively. Note that in Figure 1a the feasible region with $n = 2$ is the shaded shape, while in Figure 1b the feasible region with $n = 3$ is the shaded solid.

![Feasible Region Diagram](image)

(a) (b)

**Figure 1:** Possible configurations of the feasible region of the L1-budgeted variant when (a) $n = 2$; (b) $n = 3$.

5.2 Search space

The L1-budgeted variant is a constrained optimization problem with a non-convex objective function. Fortunately, the marginal monotonicity of the function confines the search space of the variant. The following lemma shows that it suffices to look for an L1-budgeted optimum in the set of points on the boundary of the feasible region that either lies on the L1-norm ball centred at $\alpha^{(0)}$ with radius $b$ or is an extreme point of the feasible region of the corresponding unbudgeted variant.
Lemma 5.1 (Confined Search Space). There exists an L1-budgeted optimum \( \alpha^* \) that satisfies either \( \| \alpha^* - \alpha^{(0)} \|_1 = b \) or \( \alpha^* \in \prod_{i \in [n]} \{ l_i, u_i \} \).

**Proof.** Let \( \Omega = \left\{ \alpha \in \prod_{i \in [n]} \{ l_i, u_i \} : \| \alpha - \alpha^{(0)} \|_1 \leq b \right\} \) be the feasible region of the L1-budgeted variant. We shall prove, by induction, that for any \( 1 \leq k \leq n \), if \( \alpha \in \Omega \) is an L1-budgeted optimum with \( \| \alpha - \alpha^{(0)} \|_1 < b \), \( \alpha_i \in \{ l_i, u_i \} \) for \( i \in [n] \setminus [k] \), and \( \alpha_i \notin \{ l_i, u_i \} \) for \( i \in [k] \), then there exists an L1-budgeted optimum \( \alpha^* \) that satisfies either \( \| \alpha - \alpha^{(0)} \|_1 = b \) or \( \alpha^* \in \prod_{i \in [n]} \{ l_i, u_i \} \). Then, this gives the desired result by symmetry of the agents.

In the base case, suppose that \( \alpha \in \Omega \) is an L1-budgeted optimum with \( \| \alpha - \alpha^{(0)} \|_1 < b \), \( \alpha_i \in \{ l_i, u_i \} \) for \( i \in [n] \setminus [1] \), and \( \alpha_1 \notin \{ l_1, u_1 \} \). Then \( \partial f(\alpha) / \partial \alpha_1 = 0 \). By Lemma 4.1, \( f(\alpha) = f(\alpha^*) \) for any \( \alpha^* \in \prod_{i \in [n]} \{ l_i, u_i \} \) with \( \alpha^*_i = \alpha_i \) for \( i \in [n] \setminus \{ 1 \} \). In particular, if we set \( \alpha^*_1 = \min \{ u_1, b - \sum_{i \in [n] \setminus \{ 1 \}} |\alpha_i - \alpha_i^{(0)}| \} \), then \( \alpha^* \) is another L1-budgeted optimum that satisfies either \( \| \alpha - \alpha^{(0)} \|_1 = b \) or \( \alpha^* \in \prod_{i \in [n]} \{ l_i, u_i \} \).

For the inductive step, assume the following: if \( \alpha \in \Omega \) is an L1-budgeted optimum with \( \| \alpha - \alpha^{(0)} \|_1 < b \), \( \alpha_i \in \{ l_i, u_i \} \) for \( i \in [n] \setminus [k] \), and \( \alpha_i \notin \{ l_i, u_i \} \) for \( i \in [k] \), then there exists an L1-budgeted optimum \( \alpha^* \) that satisfies either \( \| \alpha - \alpha^{(0)} \|_1 = b \) or \( \alpha^* \in \prod_{i \in [n]} \{ l_i, u_i \} \).

Now, suppose that \( \alpha \in \Omega \) is an L1-budgeted optimum with \( \| \alpha - \alpha^{(0)} \|_1 < b \), \( \alpha_i \in \{ l_i, u_i \} \) for \( i \in [n] \setminus [k+1] \), and \( \alpha_i \notin \{ l_i, u_i \} \) for \( i \in [k+1] \). Then \( \partial f(\alpha) / \partial \alpha_i = 0 \) for any \( i \in [k+1] \). In particular, \( \partial f(\alpha) / \partial \alpha_{k+1} = 0 \). By Lemma 4.1, \( f(\alpha) = f(\alpha^*) \) for any \( \alpha^* \in \prod_{i \in [n]} \{ l_i, u_i \} \) with \( \alpha^*_i = \alpha_i \) for \( i \in [n] \setminus \{ k+1 \} \).

Set \( \alpha^*_{k+1} = \min \{ u_{k+1}, b - \sum_{i \in [n] \setminus \{ k+1 \}} |\alpha_i - \alpha_i^{(0)}| \} \). If \( u_{k+1} \geq b - \sum_{i \in [n] \setminus \{ k+1 \}} |\alpha_i - \alpha_i^{(0)}| \), then \( \alpha^* \) is another L1-budgeted optimum that satisfies \( \| \alpha - \alpha^{(0)} \|_1 = b \). Otherwise, \( \alpha^*_{k+1} = u_{k+1} \), so \( \alpha^* \) is an L1-budgeted optimum with \( \| \alpha - \alpha^{(0)} \|_1 < b \), \( \alpha_i \in \{ l_i, u_i \} \) for \( i \in [n] \setminus [k] \), and \( \alpha_i \notin \{ l_i, u_i \} \) for \( i \in [k] \). By the inductive hypothesis, there exists another L1-budgeted optimum that satisfies either \( \| \alpha - \alpha^{(0)} \|_1 = b \) or \( \alpha^* \in \prod_{i \in [n]} \{ l_i, u_i \} \).

This completes the induction proof. \( \square \)
By Lemma 5.1, the search space of the L1-budgeted variant only consists of the union some faces of the feasible region and some extreme points of the corresponding unbudgeted variant. To illustrate the lemma, Figure 2 shows the two search spaces of the L1-budgeted variant under the same parameter settings as in Figure 1. Note that in Figure 2a, the search space when \( n = 2 \) consists of three line segments and an extreme point marked with crosses, while in Figure 2b, the search space when \( n = 3 \) consists of six shaded faces and an extreme point marked with a cross.

**Figure 2:** Possible configurations of the search space of the L1-budgeted variant when (a) \( n = 2 \); (b) \( n = 3 \).
6 Projected Gradient Algorithm

Based on the properties of the objective function and the search space of the L1-budgeted variant mentioned in Section 4 and 5, the Projected Gradient Algorithm is adopted to solve the L1-budgeted variant.

6.1 The main algorithm

Our Projected Gradient Algorithm, given in Algorithm 3, consists of two main parts. The first part is the Optimistic Local Search function call in line 1. This function, which is an implementation of the Optimistic Local Search algorithm proposed in Chan et al. [2], returns a corresponding unbudgeted optimum. The details of the Optimistic Local Search algorithm are in the Appendix. If the unbudgeted optimal point found is inside the L1-norm ball, it is feasible in the L1-budgeted variant. In this case, the algorithm will return the unbudgeted optimum (line 2). Otherwise, it continues to look for the L1-budgeted optimum in the search space suggested in Lemma 5.1 and the algorithm will move on to the second part.

To simplify the implementation of the second part of the algorithm, the change of variables $\alpha \rightarrow \frac{\alpha - \alpha^{(0)}}{b}$ is introduced to the L1-budgeted variant. This leads to the following minimization problem:

$$\min \hat{f}(x) := 1^T [I - (I - X)P]^{-1}Xs, \text{ where } X = \text{Diag}(\alpha^{(0)} + bx)$$

subject to

$$L_i := \max \left\{ \frac{l_i - \alpha^{(0)}_i}{b}, -1 \right\} \leq x_i \leq \min \left\{ \frac{u_i - \alpha^{(0)}_i}{b}, 1 \right\} := U_i \text{ for all } i$$

$$\|x\|_1 \leq 1$$

Each variable $x_i$ has a physical meaning: its sign represents the direction of change of the resistance parameter $\alpha_i$, whereas its magnitude represents the proportion of budget to be distributed to modify the resistance of agent $i$. For each $i$, $U_i$ and $L_i$ define the upper and lower bounds of $x_i$ respectively.

Our insight is that the L1-budgeted optimal point should not be far away from the corresponding unbudgeted optimum. Therefore, in line 3, the algorithm exploits the unbudgeted optimum returned to locate an initial point on the face of the search space such that the direction of movement of each coordinate from the initial resistance to the initial point is the same as that to the unbudgeted optimum. Based on the projected gradient computed, the algorithm will iteratively find a better point than the current one. It will return the current point if either the current point is a local minimum in the feasible region (line 7-8), or the projected gradient at the current point is too small (line 12-13).
Algorithm 3: Projected Gradient Algorithm

\textbf{Input:} Innate opinions \( s \in [0, 1]^n \); interaction matrix \( P \); for each agent \( i \in [n] \), upper \( u_i \) and lower \( l_i \) bounds for resistance, as well as initial resistance \( \alpha_i^{(0)} \); budget \( b \); termination scalar \( \varepsilon \); approximation parameters \( \delta_1, \delta_2 \).

\textbf{Output:} L1-budgeted optimal resistance vector \( \alpha \in \{ \alpha \in \prod_{i \in [n]} [l_i, u_i] : \| \alpha - \alpha^{(0)} \|_1 \leq b \} \).

1. Set \( \alpha^U \leftarrow \text{Optimistic Local Search}(s, P, u, l) \)
2. if \( \| \alpha^U - \alpha^{(0)} \|_1 \leq b \) then return \( \alpha^U \)
3. Initially, for each agent \( i \), set \( L_i \leftarrow l_i - \alpha^{(0)}_i \), \( U_i \leftarrow u_i - \alpha^{(0)}_i \), \( v_i \leftarrow \begin{cases} L_i, & \text{if } \alpha^U_i = l_i \\ U_i, & \text{if } \alpha^U_i = u_i \end{cases} \), and \( x \leftarrow \frac{v_i}{\| v_i \|_2} \).
4. while True do
5. \( d \leftarrow \text{Projected Gradient}(x, L, U, \delta_1) \)
6. if \( \exists j \in [n], c \in \mathbb{R} : d = ce_j \) then
7. \( \text{if } x_j = 0 \lor c = 0 \lor (x_j < 0 \land c > 0) \lor (x_j > 0 \land c < 0) \) then
8. \( \text{return } \alpha^{(0)} + bx \)
9. else
10. \( x_j \leftarrow \begin{cases} \min\{-x_j, U_j\}, & \text{if } x_j < 0 \\ \max\{-x_j, L_j\}, & \text{otherwise} \end{cases} \)
11. \( f^* \leftarrow \text{Objective Approximation}(\alpha^{(0)} + bx, s, P, \delta_2) \)
12. else if \( \forall i \in [n] : d_i < \varepsilon \) then
13. \( \text{return } \alpha^{(0)} + bx \)
14. else
15. \( x \leftarrow \text{Stepping}(x, d, f^*, L, U, \delta_2) \)
16. \( f^* \leftarrow \text{Objective Approximation}(\alpha^{(0)} + bx, s, P, \delta_2) \)

6.2 Determining the projected gradient

The search space of the L1-budgeted variant consists of multiple faces and points on the boundary of the feasible region. Our algorithm ensures that the point obtained after each iteration remains in the search space by carefully choosing the direction of movement and the step size.

Algorithm 4 handles the direction of movement. At the current point, the algorithm first approximates the gradient of the objective function in line 1 by calling Algorithm 2. Then in line 2, the algorithm will determine an initial face for gradient projection according to the current point and its gradient, where the vector \( q \) is the face indicating vector. After that, Algorithm 5 is called in line 3 to generate the desired projection matrix \( H \). From line 6 to line 9, the algorithm will repeatedly review and modify the face for gradient projection until either the projected gradient points into the search space or the gradient is projected onto a 1-dimensional subspace.
Algorithm 4: Projected Gradient

**Input:** Current budget distribution vector \( x \in [0, 1]^n \); for each agent \( i \in [n] \), upper \( U_i \) and lower \( L_i \) bounds for budget distribution; approximation parameter \( \delta \).

**Output:** Projected gradient vector \( d \in \mathbb{R}^n \).

1. \( g \leftarrow \text{Gradient Approximation}(x(0) + bx, s, P, \delta) \)
2. \( q \leftarrow \begin{cases} -1, & \text{if } x < 0 \lor (x = 0 \land g_i \geq 0) \\ 1, & \text{otherwise} \end{cases} \)
3. \( H \leftarrow \text{Projection matrix}(q) \)
4. \( d \leftarrow Hg \)
5. \( \text{Denote } E := \{i \in [n] : q_i \neq 0 \land (x_i = L_i \land d_i > 0) \lor (x_i = 0 \land q_i = 1 \land d_i > 0) \lor (x_i = U_i \land d_i < 0) \lor (x_i = 0 \land q_i = -1 \land d_i < 0) \} \)
6. \( \text{while } E \neq \emptyset \land |\{i \in [n] : q_i = 0\}| > 1 \) do
7. \( \text{Pick an arbitrary } j \in E \text{ and set } q_j \leftarrow 0 \)
8. \( H \leftarrow \text{Projection matrix}(q) \)
9. \( d \leftarrow Hg \)
10. \( \text{return } \text{Projected gradient vector } d. \)

Algorithm 5: Projection matrix

**Input:** Face indicating vector \( q \).

**Output:** Projection matrix \( H \in \{-1, 0, 1\}^{n \times n} \).

1. Denote \( Z := \{i \in [n] : q_i \neq 0\} \).
2. Initialize \( H \leftarrow -\frac{1}{|Z|}qq^T \)
3. \( \text{if } |Z| > 1 \) then
   4. \( \text{for each } i \in Z \) do
      5. \( H_{ii} \leftarrow H_{ii} + 1 \)
6. \( \text{return } \text{Projection matrix } H. \)

The generation of projection matrix \( H \) of a given face indicating vector \( q \in \mathbb{R}^n \) in Algorithm 5 will be justified below. The set \( \{i \in [n] : q_i \neq 0\} \) will be denoted by \( Z \).

If \( |Z| = 1 \) and the only nonzero component in \( q \) is \( q_i \), then according to the algorithm, \( H \) is the \( n \times n \) matrix with the only nonzero entry \( H_{ii} = 1 \), which is the desired projection matrix because the only gradient component to be projected in this case is the \( i \)-th component.

Suppose \( |Z| > 1 \). Pick an arbitrary \( k \in [n] \) such that \( q_k \neq 0 \). Define \( C \in \mathbb{R}^{n \times (|Z|-1)} \) such that each column is equal to \( q_je_i - q_je_k \) for some \( i \in Z \) with \( i \neq j \) and no two columns are equal. Then \( C \) is the design matrix of the projection. We will show that the matrix \( H \) generated in the algorithm is equal to \( U := C(C^TC)^{-1}C^T \).

First, for any \( i \notin Z \), \( H_{ij} = H_{ji} = 0 \) for any \( j \in [n] \) because \( q_i = 0 \). Also, \( U_{ji} = U_{jj} = 0 \) for any \( j \in [n] \) because the \( i \)-th row of \( U \) is zero.
Second, for any distinct \(i, j \in \mathbb{Z}\), \(H_{ij} = -\frac{q_i q_j}{|\mathbb{Z}|}\). To evaluate \(U_{ij}\), note that \((C^T C)_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}\)

i.e. \(C^T C = I + 11^T \in \mathbb{R}^{(|\mathbb{Z}|^{-1}) \times (|\mathbb{Z}|^{-1})}\). Hence, by Sherman-Morrison formula, we have

\[
(C^T C)^{-1} = (I + 11^T)^{-1} = I^{-1} - \frac{I^{-1}11^TI^{-1}}{1 + 1^TI^{-1}1} = I - \frac{J}{1 + |\mathbb{Z}|^{-1}} = I - \frac{J}{|\mathbb{Z}|}.
\]

If \(i \neq k\) and \(j \neq k\), \(U_{ij} = (q_i e_i)^T (C^T C)^{-1} (q_j e_j) = q_i q_j e_i^T (I - \frac{J}{|\mathbb{Z}|}) e_j = -\frac{q_i q_j}{|\mathbb{Z}|} e_j\).

If \(i = k\), \(U_{ij} = q_i 1^T (C^T C)^{-1} (q_j e_j) = q_i q_j 1^T (I - \frac{J}{|\mathbb{Z}|}) e_j = q_i q_j \left(1 - \frac{|\mathbb{Z}| - 1}{|\mathbb{Z}|}\right) = -\frac{q_i q_j}{|\mathbb{Z}|} e_j\).

If \(j = k\), \(U_{ij} = q_i e_i^T (C^T C)^{-1} (q_j 1) = q_i q_j e_i^T (I - \frac{J}{|\mathbb{Z}|}) 1 = q_i q_j \left(1 - \frac{|\mathbb{Z}| - 1}{|\mathbb{Z}|}\right) = -\frac{q_i q_j}{|\mathbb{Z}|} e_i\).

Finally, for any \(i \in \mathbb{Z}\), \(H_{ii} = 1 - \frac{1}{|\mathbb{Z}|} = e_i^T (C^T C)^{-1} e_i = U_{ii}\).

Therefore, \(H = U\). With the above justification, the direct evaluation of \(U\), which involves the matrix inverse \((C^T C)^{-1}\), can be avoided. This helps reduce the computational time of the algorithm.

### 6.3 Choosing the step size

If the gradient is projected onto a 1-dimensional subspace, then the main algorithm will either terminate or set the next point at an extreme point of the unbudgeted feasible region. This decision depends on whether the current point is a local minimum (line 7-11 of Algorithm 3). Otherwise, the algorithm will choose a step size depending on the current point and the projected gradient. The step size, denoted by \(h\) in Algorithm 6, is first set to be the largest possible to hit either the lower bound or the upper bound of some \(x_i\) on the face of projection (line 1-3). The algorithm will then adjust the step size to ensure that the next point has a lower objective value than the current point (line 4-7).
**Algorithm 6: Stepping**

**Input:** Current budget distribution vector \( x \in [0,1]^{n} \); projected gradient \( d \); objective value at the current point \( f^* \in [0,1] \); for each agent \( i \in [n] \), upper \( U_i \) and lower \( L_i \) bounds for budget distribution; approximation parameter \( \delta \).

**Output:** Next budget distribution vector \( x' \in [0,1]^{n} \).

1. For each agent \( i \), set \( v_i \leftarrow \begin{cases} L_i, & \text{if } x_i \leq 0 \land d_i > 0 \\ U_i, & \text{if } x_i \geq 0 \land d_i < 0 \\ 0, & \text{otherwise} \end{cases} \)

2. \( r \leftarrow \max \left\{ \frac{d_i}{v_i - x_i} : i \in [n], v_i - x_i \neq 0 \right\} \)

3. \( h \leftarrow \frac{1}{r} \)

4. \( f' \leftarrow \text{Objective Approximation} \left( \alpha^{(0)} + b(x - h), s, P, \delta \right) \)

5. while \( f' > f^* \) do

6. \( h \leftarrow \frac{h}{2} \)

7. \( f' \leftarrow \text{Objective Approximation} \left( \alpha^{(0)} + b(x - h), s, P, \delta \right) \)

8. return Next budget distribution vector \( x' = x - h \).

The initial step size is set as the largest possible to raise the efficiency of the algorithm. Figure 3 shows the visualization of an example run of the algorithm with \( n = 3 \). To better understand the objective function, random sampling is carried out in the search space. With a smaller step size (see Figure 3a), no obvious twist is observed on the search paths except when they hit on an edge. Nevertheless, with the largest possible step size for the same problem (see Figure 3b), the L1-budgeted optimum is reached in 2 steps. Hence the choice of step size.

**Figure 3:** A sample run of the projected gradient algorithm when \( n = 3 \) (a) with a small step size; (b) with the largest step size possible.
7 Experiments

The Projected Gradient Algorithm has been implemented in Python. The codes are available on GitHub at [https://github.com/tracylcs/opinion-dynamics](https://github.com/tracylcs/opinion-dynamics). To test the algorithm, the variant parameters $s$, $P$, $l$, $u$, $\alpha^{(0)}$, and $b$ are drawn to form random instances of the L1-budgeted variants. In particular, the algorithm runs on small networks (say, $n = 2$ or $n = 3$) are used to understand the searching process, while those on large networks are for investigating the algorithm scalability.

All our experiments run on a mobile computer with up to 3.90 GHz Intel Core i7-1065G7 CPU, 16 GB main memory, and 8 threads.

7.1 Sample runs on small networks

In all the figures in the rest of the report, we shall use the same set of markers as in Figure 3 of the last section to represent the search path, the initial point where the algorithm starts, and the best point yielded from random sampling. Figures 4 and 5 show the visualization of some sample runs of the algorithm on networks with 2 and 3 agents respectively.

![Figure 4](image)

**Figure 4:** Three sample runs of the projected gradient algorithm when $n = 2$

![Figure 5](image)

**Figure 5:** Three sample runs of the projected gradient algorithm when $n = 3$

In all cases shown in Figure 4, the L1-budgeted optimal point is reached in one step. Moreover, exactly one of $x_1, x_2$ equals its upper or lower bound. Indeed, we have $x_1 = L_1$ in Figure 4a, $x_2 = L_2$ in Figure 4b, and $x_2 = U_2$ in Figure 4c. When $n = 3$, two steps are required in general to reach the L1-budgeted optimal point. In all cases shown in Figure 5, exactly two of $x_1, x_2$, and $x_3$ equal its upper or lower bound.
7.2 Sample runs on larger networks

To test the algorithm scalability, three instances of the L1-budgeted variant with 100, 200, and 500 agents are created. For each instance, we ran the main algorithm with $\varepsilon = 10^{-14}$ and $\delta_1 = \log n$ and $\delta_2 = 0.1$.

Table 1 shows the details of the datasets and the information about the algorithm runs.

<table>
<thead>
<tr>
<th>Artificial Dataset</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of nodes ($n$)</td>
<td>100</td>
<td>200</td>
<td>500</td>
</tr>
<tr>
<td>No. of edges</td>
<td>4961</td>
<td>20093</td>
<td>124750</td>
</tr>
<tr>
<td>Proportion of budget spent by the solution returned</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Lying on the face which the algorithm starts?</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>No. of coordinates $x_i$ of the solution $x$ returned not equal to $0 / L_i / U_i$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Running time</td>
<td>10.54 s</td>
<td>23.53 s</td>
<td>22.00 min</td>
</tr>
<tr>
<td>No. of iterations</td>
<td>99</td>
<td>212</td>
<td>533</td>
</tr>
<tr>
<td>Average time of an iteration</td>
<td>0.106 s</td>
<td>0.111 s</td>
<td>2.476 s</td>
</tr>
</tbody>
</table>

Table 1 suggests that in all the three artificial datasets, the optimal solution returned by the Projected Gradient Algorithm always uses up the budget. Moreover, it lies on the face which the algorithm starts. This indicates that choosing an initial point on the $n - 1$ face that faces the unbudgeted optimum should be a good decision. Furthermore, all except one coordinate of the solution $x$ returned equals either 0 or one of its bounds. This means all the solutions returned are a vertex of the feasible region.

7.3 Limitations of the algorithm

The running time of the algorithm is not satisfactory. Table 1 suggests that in all cases, the number of iterations is slightly more than $n$. However, the running time per iteration upsurges as $n$ increases, leading to a total running time of around 22 minutes for Dataset C with 500 agents. The major barrier is that as the number of agents increases, the number of modification required to identify the face for gradient projection increases significantly (line 6-9 of Algorithm 4). The problem could possibly be solved using a similar approach as the Batch Gradient Greedy algorithm in [12], where a gradient vector is used for directing the descent for some number of iterations to reduce the computational time. Due to the project duration, the approach is not considered in this project. However, it is also noted that the Batch Gradient Greedy approach may increase the optimality gap of the algorithm. The actual trade-off worths further investigation.
8 Conclusion

The Opinion Susceptibility Problem explores the optimal way to interfere the public opinions in a social network via a modification of the level of susceptibility to persuasion of the agents inside. Based on the opinion dynamics model proposed by Abebe et al. [1], this project studies the L1-budgeted variant of the problem to address the cost of resistance intervention in reality. In this problem, a perturbation budget is given, which bounds the total change of susceptibility to persuasion (or the resistance parameter) of the agents in the network. It asks for the optimal way to distribute the budget so that the sum of equilibrium opinions is minimized.

After Abebe et al. [1] and Chan et al. [2] proved that the L0-budgeted variant is NP-hard and the unbudgeted variant is in P respectively, this project establishes the NP-hardness of the L1-budgeted problem by showing that the vertex cover problem for regular graphs is polynomial-time reducible to the L1-budgeted problem. Hence, the project also develops an algorithm that approximates the optimal solution to the L1-budgeted problem. The marginal monotonicity of the objective function guarantees the existence of an L1-budgeted optimum in a subset of the feasible region boundary. Based on the findings, the Projected Gradient Algorithm is constructed to locate an optimal point only in the subset. The Optimistic Local Search method is employed to locate a starting point of the algorithm near the unbudgeted optimum. It is found by experiment that this starting point is always in the same direction as the optimal point returned with respect to the initial resistance. The objective and the gradient approximations by iterative processes also allow an efficient evaluation of optimality in the algorithm. Experiments on some randomly generated instances of the L1-budgeted variants reveal that the optimal point is usually a vertex of the feasible region that tends to use up the budget.

Given the NP-hardness of the L1-budgeted problem, a natural further question would be the hardness of approximation of the problem. This problem shall be left as future work.
References


Algorithm 7: Optimistic Local Search (Proposed by Chan et al. [2])

Input: Innate opinions $s \in [0, 1]^n$; interaction matrix $P$; for each agent $i \in [n]$, upper $u_i$ and lower $l_i$ bounds for resistance.

Output: Unbudgeted optimal resistance vector $\alpha \in \prod_{i \in [n]} \{l_i, u_i\}$.

1 (Technical step.) Randomly perturb each coordinate of $s$ slightly.
2 Initially, for each agent $i$, set $\alpha_i \leftarrow u_i$ to its upper bound; denote $\varepsilon_\alpha := \min_{i \in [n]} \alpha_i$.
3 Pick $z = 1$, and set $t \leftarrow 0$; denote $\text{err}(t) := \frac{(1 - \varepsilon_\alpha)^t}{\varepsilon_\alpha}$.
4 while $\exists i \in [n] : |s_i - z_i| \leq \text{err}(t)$ do
5 $z \leftarrow As + (I - A)Pz$, where $A = \text{Diag}(\alpha)$
6 $t \leftarrow t + 1$
7 (Optimistic Candidates.) Set $L \leftarrow \{i \in [n] : z_i \leq s_i \land \alpha_i = u_i\}$.
8 (Rare Mistakes.) Set $J \leftarrow \{i \in [n] : z_i > s_i \lor \alpha_i = l_i\}$.
9 if $L \cup J \neq \emptyset$ then
10 for each $i \in L$ do
11 Set $\alpha_i \leftarrow l_i$ to its lower bound (and update $\varepsilon_\alpha$).
12 for each $i \in J$ do
13 Set $\alpha_i \leftarrow u_i$ to its upper bound (and update $\varepsilon_\alpha$).
14 $t \leftarrow 0$
15 return Resistance vector $\alpha$. 

Appendix: Optimistic Local Search